very bad. The non-defective cases, with several vectors nearly parallel, were sometimes worse than the defective cases.

\(N = 6\) \(s = -4\)

actual roots

\[0, 0, 2, 2, -1, -4, -10, -18\]

approximations (7070)

\[0.0228, -0.0225, 1.995, 2.004, -1, -4, -10, -18\]

\(N = 6\) \(s = -6\)

actual roots

\[-6, 0, 0, 4, 4, 6, 6\]

approximations (7070)

\[-6, -0.0056 \pm .111i, 3.969 \pm .363i, 6.036 \pm .2215i\]

\(N = 6\) \(s = -6.5\)

actual roots

\[-3, 0, 2.5, 4.5, 6, 7, 7.75\]

approximations (7070)

\[-3, .0066, 2.436, 5.103, 5.151, 7.401 \pm .405i\]

The effect of higher precision is seen in the following example which was run on both the 7070 (8 digits) and the 1604 (10 digits)

\(N = 10\) \(s = -14\)

actual roots

\[0, -2 \times 10^{-7}, 0, 0, 12, 12.00004, 11.898, 22, 21.995, 20.195, 30, 30, 29.638 \pm 1.134i, 23.843 \pm 7.16i, 36, 36, 34.993 \pm 2.94i, 33.926 \pm 12.92i, 40, 40, 40.024 \pm 3.39i, 36.977, 42, 42, 43.347 \pm 1.49i, 46.387 \pm 10.99i\]

approximations (CDC 1604)

\[0.0026, 11.898, 20.195, 23.843 \pm 7.16i, 33.926 \pm 12.92i, 36.977, 46.387 \pm 10.99i\]

approximations (IBM 7070)

\[0.0026, 11.898, 20.195, 23.843 \pm 7.16i, 33.926 \pm 12.92i, 36.977, 46.387 \pm 10.99i\]

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Multivariate Polynomial Approximation for Equidistant Data

By B. Mond

Abstract. The theory of polynomial approximation for evenly spaced points is extended to multivariate polynomial approximation. It is also shown how available tables prepared for univariate approximation can be used in the multivariate case.

1. Introduction. Assume \(f(x)\) is given for \(x = x_1, x_2, \cdots, x_n\) and it is desired to approximate \(f(x)\) by a polynomial of degree \(p\), \(1 \leq p < n\), i.e.

\[f(x) \approx \sum_{i=0}^{p} a_ix^i.\]

Received November 12, 1963.
In order to determine the coefficients \( a_i \) so as to minimize
\[
\sum_{j=1}^{n} \left[ f(x_j) - \sum_{i=0}^{p} a_i x_j^i \right]^2
\]
one must solve a set of \( p + 1 \) normal equations. If the \( x_i \) are evenly spaced, the
problem can be greatly simplified by a change of variable and the use of orthogonal
polynomials [4]. This simplification will now be extended to approximation by
multivariate polynomials.

### 2. Bivariate Polynomial Approximation

Assume that \( f(x, y) \) is defined on a
finite planar set of points \( \{(x_i, y_j)\} (i = 1, 2, \cdots, n; j = 1, 2, \cdots, m) \) and that it
is desired to approximate \( f(x, y) \) by a bivariate polynomial of the form
\[
\sum_{h=0}^{s} \sum_{k=0}^{r} a_{kh} x^k y^h
\]
\( 1 \leq r < n \) and \( 1 \leq s < m \). In order to determine the coefficients \( a_{kh} \) \((k = 0, 1, \cdots, r; h = 0, 1, \cdots, s) \) so as to minimize
\[
\sum_{j=1}^{m} \sum_{i=1}^{n} \left[ f(x_i, y_j) - \sum_{h=0}^{s} \sum_{k=0}^{r} a_{kh} x_i^k y_j^h \right]^2
\]
one must solve a system of \((r + 1)(s + 1)\) normal equations.

Assume, now, that the \( x_i \) and \( y_j \) are evenly spaced, i.e.
\[
x_{k+1} - x_k = h_1 \quad (k = 1, 2, \cdots, n - 1)
\]
\[
y_{k+1} - y_k = h_2 \quad (k = 1, 2, \cdots, m - 1).
\]
Thus, if we write
\[
(x, y) = (a + ih_1, b + jh_2),
\]
where \( a = x_1 - h_1 \) and \( b = y_1 - h_2 \), \((x, y)\) will represent the coordinates of the
given points when \( i = 1, \cdots, n; j = 1, \cdots, m \). Let \( \hat{x} \) and \( \hat{y} \) be the midpoints of
the given \( x_i \) and \( y_j \) by
\[
x_i = \frac{x_1 + x_n}{2}; \quad y_j = \frac{y_1 + y_m}{2}.
\]
The horizontal and vertical

distance from the midpoint divided by the uniform spacing \([[x - \hat{x}]/h_1 \quad (y - \hat{y})]/h_2 \) are then equal respectively, by virtue of equations (1), to
\[
i - (n + 1)/2 \quad j - (m + 1)/2.
\]
Let \( P_{hk} \) \((h = 0, 1, \cdots, r; k = 0, 1, \cdots, s) \) be a set of polynomials of exact
degree \( r \) and \( s \) in \( i - (n + 1)/2 \) and \( j - (m + 1)/2 \), i.e.
\[
P_{hk} = \alpha_{00} + \alpha_{10} i - (n + 1)/2 \] \( + \alpha_{01} j - (m + 1)/2 \) \( + \alpha_{11} i - (n + 1)/2 \] \( + \alpha_{02} j - (m + 1)/2 \] \( + \alpha_{12} i - (n + 1)/2 \] \( + \alpha_{22} j - (m + 1)/2 \] \[ + \cdots \]
\[
P_{00} \quad (2)
\]
\( P_{00} \) will always be taken equal to one.

If, now, we approximate \( f(x, y) \) by a polynomial of the form
\[
\sum_{h=0}^{s} \sum_{k=0}^{r} b_{kh} P_{hk}
\]
and solve for the constants $b_{hk}$ in the sense of least squares, we obtain $(r + 1)$ $(s + 1)$ normal equations with the augmented matrix

$$
\begin{bmatrix}
\sum \sum P_{00} & \sum \sum P_{01} \cdots & \sum \sum P_{0s} & \sum \sum P_{10} \cdots & \sum \sum P_{1s} & \cdots & \sum \sum f(x_i, y_j) \\
\sum \sum P_{00} P_{00} & \sum \sum P_{01} P_{00} \cdots & \sum \sum P_{0s} P_{00} & \sum \sum P_{10} P_{00} \cdots & \sum \sum P_{1s} P_{00} & \cdots & \sum \sum f(x_i, y_j) P_{00} \\
& \vdots & & & & & \\
\sum \sum P_{00} P_{s0} & \sum \sum P_{01} P_{s0} \cdots & \sum \sum P_{0s} P_{s0} & \sum \sum P_{10} P_{s0} \cdots & \sum \sum P_{1s} P_{s0} & \cdots & \sum \sum f(x_i, y_j) P_{s0} \\
\sum \sum P_{00} P_{rs} & \sum \sum P_{01} P_{rs} \cdots & \sum \sum P_{0s} P_{rs} & \sum \sum P_{10} P_{rs} \cdots & \sum \sum P_{1s} P_{rs} & \cdots & \sum \sum f(x_i, y_j) P_{rs}
\end{bmatrix}
$$

where the notation $\sum \sum P_{hk} P_{cd}$ means

$$
\sum_{j=1}^{m} \sum_{i=1}^{n} P_{hk}[i - (n + 1)/2, j - (m + 1)/2]P_{cd}[i - (n + 1)/2, j - (m + 1)/2].
$$

If the $a_{ij}$ in (2) are chosen so that the polynomials $P_{hk}$ are biorthogonal, i.e.

$$
\sum_{j=1}^{m} \sum_{i=1}^{n} P_{hk} P_{cd} = 0 \quad \text{if } h \neq c \text{ or } k \neq d,
$$

the coefficient matrix reduces to a diagonal matrix. The $b_{hk}$ can then be written as

$$
b_{hk} = \sum_{j} \sum_{i} f(x_i, y_j) P_{hk}/\sum_{j} \sum_{i} P_{hk}^2.
$$

As in the univariate case [2], testing the appropriateness of the representation is also facilitated by the use of orthogonal polynomials. Should a polynomial of higher degree be desired, the coefficients $b_{hk}$ need not be recalculated.

3. Constructing Biorthogonal Polynomial Tables. In approximating with orthogonal functions, many of the calculations necessary are independent of the data [for example, the denominator in (3)] and need not be recalculated each time a different set of data is to be fitted. Extensive tables are available for univariate orthogonal polynomials [1], [2]. Their use makes the actual calculation of the coefficients quite easy.

By appropriately choosing the bivariate biorthogonal polynomials, univariate tables that are already available can easily be modified for use in the bivariate case.

**Theorem.** Let $P_{h}$ ($h = 0, 1, \ldots , r$) and $P_{k}$ ($k = 0, 1, \ldots , s$) be sets of orthogonal polynomials in $(i - (n + 1)/2)$ and $(j - (m + 1)/2)$ respectively with $P_{00} = 1$. Let $P_{hk} = P_{h} P_{k}$. $P_{hk}$ ($h = 0, 1, \ldots , r; k = 0, 1, \ldots , s$) is then a set of bivariate biorthogonal polynomials in $(i - (n + 1)/2)$ and $(j - (m + 1)/2)$.

**Proof.** It follows from the definition that $P_{hk}$ will be a bivariate polynomial of degree $h$ and $k$ in $(i - (n + 1)/2)$ and $(j - (m + 1)/2)$ respectively.

Biorthogonality of the bivariate polynomials follows from the orthogonality of the univariate polynomials and the fact that

$$
\sum_{j} \sum_{i} P_{hk} P_{cd} = \sum_{j} \sum_{i} P_{h} P_{k} P_{c} P_{d} = \sum_{j} P_{h} P_{c} \sum_{i} P_{k} P_{d}.
$$

Taking $P_{hk} = P_{h} P_{k}$, it is possible to utilize, in bivariate polynomial approximation, tables that were constructed for use in the univariate case. In general, if one regards the rectangular array of values for a given $n$ in Fisher and Yates statistical tables [2] as a matrix, then the corresponding matrix for bivariate biorthogonal polynomials would be a Kronecker product [5] of corresponding matrices.
For example, from the entries for $n = 3$ and $n = 4$ in the Fisher and Yates tables [2], one gets as the entry in the bivariate table for $n = 3, m = 4$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$j$</th>
<th>$P_{01}$</th>
<th>$P_{02}$</th>
<th>$P_{03}$</th>
<th>$P_{10}$</th>
<th>$P_{11}$</th>
<th>$P_{12}$</th>
<th>$P_{13}$</th>
<th>$P_{20}$</th>
<th>$P_{21}$</th>
<th>$P_{22}$</th>
<th>$P_{23}$</th>
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<td>-1</td>
<td>-1</td>
<td>+3</td>
<td>-1</td>
<td>+1</td>
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<td>-3</td>
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<td>-1</td>
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<tr>
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<td>-1</td>
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<td>+3</td>
<td>+1</td>
<td>+1</td>
<td>+1</td>
</tr>
</tbody>
</table>

| 60 | 12 | 60 | 8 | 40 | 8 | 40 | 24 | 120 | 24 | 120 |

where the last line represents the sums of squares of all elements in the particular column. All entries in the bivariate table are products of corresponding entries in the univariate table.

Bivariate tables constructed as outlined here could be used in exactly the same manner as recommended for univariate tables [see 2].

4. Extension to $n$ Variables. The extension to $n$ variables is straightforward. The $n$ dimension analogue to equation (3) is

$$b_{h_1...h_n} = \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} f(x_{i_1}^{(1)}, \ldots, x_{i_n}^{(n)}) P_{h_1...h_n} \sqrt{\sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} P_{h_1...h_n}^2}$$

where the $P_{h_1...h_n}$ are orthogonal polynomials in $n$ variables and the $b_{h_1...h_n}$ are the coefficients in the approximation

$$f(x^{(1)}, \ldots, x^{(n)}) \approx \sum_{h_n=0}^{r_n} \cdots \sum_{h_1=0}^{r_1} b_{h_1...h_n} P_{h_1...h_n}$$

so as to minimize

$$\sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} \left[ f(x_{i_1}^{(1)}, \ldots, x_{i_n}^{(n)}) - \sum_{h_n=0}^{r_n} \cdots \sum_{h_1=0}^{r_1} b_{h_1...h_n} P_{h_1...h_n} \right]^2.$$