3. Acknowledgments. The author gratefully acknowledges the suggestions of Mr. Charles R. Newman for programming the computer.

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### Polylogarithms, Dirichlet Series, and Certain Constants

By Daniel Shanks

The *polylogarithms* $F_s(z)$ are defined by

\[ F_s(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^s} \]

for $|z| < 1$ and for the real part of $s \geq 0$, and by analytic continuation for other values of $z$ and $s$. They can be regarded as functions of $z$, with a parameter $s$, given by the power series (1), or as functions of $s$, with a parameter $z$, given by the Dirichlet series (1).

Recently [1] we discussed the Dirichlet series defined by

\[ L_a(s) = \sum_{k=0}^{\infty} \frac{(-a)}{(2k+1)^s} \]

and its analytic continuation, where $\left( \frac{-a}{2k+1} \right)$ is the Jacobi symbol. It is expressible in closed form for three-quarters of all combinations of integers $a$ and $s$; namely, for $s \leq 1$ and all $a$, for $s$ even and $> 1$ if $a < 0$, and for $s$ odd and $> 1$ if $a > 0$.

The remaining, non-closed form $L_a(n)$ for $a = \pm 2, \pm 3, \pm 6$, with $n \leq 10$, were computed [1] by a device, which (in essence) is based on the fact that all of the so-called *characters* modulo 8, 12, or 24 are real. In contrast, the corresponding $L_a(n)$ for $a = \pm 5, \pm 7, \pm 10$, say, which were also desired, are not obtainable by that method, unless it is modified, since now some of the characters are complex.

We did, however, express $L_a(s)$ as a linear combination of the functions $S_s(x)$ or $C_s(x)$ for various values of $x$ determined by the integer $a$ [1, equations (24)–(27)]. These functions [1, equation (18)] are defined by

\[ S_s(x) = \sum_{k=0}^{\infty} \frac{\sin 2\pi(2k + 1)x}{(2k + 1)^s}, \]

\[ C_s(x) = \sum_{k=0}^{\infty} \frac{\cos 2\pi(2k + 1)x}{(2k + 1)^s}. \]

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Now consider the real and imaginary parts of $F_s(e^{i \alpha/2})$. We will call them

$$R_s(\alpha) = \Re F_s(e^{i \alpha/2})$$
$$I_s(\alpha) = \Im F_s(e^{i \alpha/2}).$$

(4)

It follows that

$$C_s(x) = R_s(4x) - \frac{1}{2^s} R_s(8x)$$
$$S_s(x) = I_s(4x) - \frac{1}{2^s} I_s(8x).$$

(5)

By the aforementioned linear combinations we may, therefore, express $L_n(s)$ in terms of the special polylogarithms (4). For example, we have

$$L_6(s) = \frac{2}{\sqrt{3}} \left[ I_s(0.2) - \frac{1}{2^s} I_s(0.4) + I_s(0.6) - \frac{1}{2^s} I_s(1.2) \right],$$

(6)

$$L_{-6}(s) = \frac{2}{\sqrt{3}} \left[ 1 + \frac{1}{2^s} \right] [R_s(0.8) - R_s(1.6)],$$

$$L_{10}(s) = \frac{2}{\sqrt{10}} \left[ I_s(0.1) - \frac{1}{2^s} I_s(0.2) - I_s(0.3) + \frac{1}{2^s} I_s(0.6) + I_s(0.7) + I_s(0.9) - \frac{1}{2^s} I_s(1.4) - \frac{1}{2^s} I_s(1.8) \right],$$

$$L_{-10}(s) = \frac{2}{\sqrt{10}} \left[ R_s(0.1) - \frac{1}{2^s} R_s(0.2) + R_s(0.3) - \frac{1}{2^s} R_s(0.6) - R_s(0.7) + R_s(0.9) + \frac{1}{2^s} R_s(1.4) - \frac{1}{2^s} R_s(1.8) \right].$$

The Computation Staff of the Amsterdam Mathematisch Centrum, under the direction of Dr. A. van Wijngaarden, has computed [2] several tables of polylogarithms accurate to 10D. Their Table III gives $R_s(\alpha)$ and $I_s(\alpha)$ for $s = 1(1)12$ and $\alpha = 0(0.01)2$. The numbers on the right side of (6) for integral $s$ are therefore given explicitly in this table, and thus, with some simple arithmetic, we obtain our Table 1.

<table>
<thead>
<tr>
<th>$s$</th>
<th>$L_6(s)$</th>
<th>$L_{-6}(s)$</th>
<th>$L_{10}(s)$</th>
<th>$L_{-10}(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.404962946</td>
<td>0.6456134114</td>
<td>0.9934588266</td>
<td>1.150086523</td>
</tr>
<tr>
<td>2</td>
<td>1.128043325</td>
<td>0.8827642541</td>
<td>0.931284985</td>
<td>1.092365033</td>
</tr>
<tr>
<td>3</td>
<td>0.839982136</td>
<td>0.9616778624</td>
<td>0.9682482537</td>
<td>1.034721928</td>
</tr>
<tr>
<td>4</td>
<td>0.102801468</td>
<td>0.9874205162</td>
<td>0.9883161275</td>
<td>1.01201984</td>
</tr>
<tr>
<td>5</td>
<td>0.100182100</td>
<td>0.9958455012</td>
<td>0.9959695576</td>
<td>1.004067704</td>
</tr>
<tr>
<td>6</td>
<td>0.100138130</td>
<td>0.9986219811</td>
<td>0.9986393802</td>
<td>1.001364688</td>
</tr>
<tr>
<td>7</td>
<td>0.100045860</td>
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<td>0.9995442414</td>
<td>1.000456202</td>
</tr>
<tr>
<td>8</td>
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<td>0.9998474373</td>
<td>0.9998477867</td>
<td>1.000152262</td>
</tr>
<tr>
<td>9</td>
<td>0.100005083</td>
<td>0.9999491729</td>
<td>0.9999492226</td>
<td>1.000050783</td>
</tr>
<tr>
<td>10</td>
<td>0.100001693</td>
<td>0.9999830616</td>
<td>0.9999830687</td>
<td>1.000016932</td>
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</tbody>
</table>

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The Hardy-Littlewood Constants

<table>
<thead>
<tr>
<th>h_{-10}</th>
<th>h_{-9}</th>
<th>h_{-8}</th>
<th>h_{-7}</th>
<th>h_{-6}</th>
<th>h_{-5}</th>
<th>h_{-4}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.67111392</td>
<td>0</td>
<td>1.85005441</td>
<td>0.75737123</td>
<td>1.38342429</td>
<td>1.77330507</td>
<td>0</td>
</tr>
</tbody>
</table>

From Table 1, in turn, we may compute [3], [4] the Hardy-Littlewood constants $h_a$ for $a = \pm 5$ and $\pm 10$. Together with previously computed values, we may thus complete an 8D table of $h_a$ for $a = -10(1)10$ except for $a = \pm 7$. The $L_{\pm 7}(s)$, needed to fill this gap, may also be expressed in terms of $I_\alpha(a)$ and $R_\alpha(a)$, but this time the arguments $\alpha$ are not given explicitly in [2], and elaborate interpolation would be required to obtain comparable precision.

Alternatively, as is known, generalized harmonic series, including $L_\alpha(s)$ for integer $s$, may be expressed in terms of the polygamma functions [5], [6]. However, the same difficulty arises for $L_{\pm 7}(s)$, and again elaborate and laborious interpolation is necessary. At the author’s request John W. Wrench, Jr. has kindly computed $L_7(2)$, $L_7(4)$, $L_{-7}(3)$ and $L_{-7}(5)$ in this way, and these numbers, together with the closed-form $L_{\pm 7}(s)$, suffice to complete our tabulation of $h_a$. This is given in Table 2.

New Factors of Fermat Numbers

By Claude P. Wrathall

Eleven new factors of Fermat numbers $F_m = 2^{2^m} + 1$ are listed below. A summary of the present status of the sequence $F_m$ is presented in Table 2.

The method used was suggested by Dr. J. L. Selfridge. Simply stated, the method consisted of forming a sieve array to eliminate possible factors divisible by a prime $\leq 499$. The remaining possible factors were tested to determine if any of the congruence relationships

[Table 2]

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