Vector Partitions and Combinatorial Identities

By M. S. Cheema

In this note we show how certain relations between vector partition functions can be deduced from certain identities. A relation connecting vector partitions having odd components and those having distinct parts will be proved. A combinatorial proof of Jacobi’s Identity similar to Franklin’s proof of Euler identity is suggested. The last section includes numerical values of $P_r(n, m)$ and $q_r(n, m)$. These results suggest the unique maxima property of $P_r(n, m)$ for fixed $n, m$ and $r$ varying.

In the Jacobi Identity

$$
\prod_{n=1}^{\infty} \left( 1 - q^{2n} \right) \left( 1 + q^{2n-1} \right) \left( 1 + q^{2n-1} t \right) = \sum_{n=0}^{\infty} q^n t^n
$$

make the substitution $q^2 = xy, t^2 = x/y$ and change $x$ to $-x$, $y$ to $-y$ to obtain

$$
\prod_{n=1}^{\infty} \left( 1 - x^n y^n \right) \left( 1 - x^n y^{-n} \right) \left( 1 - x^{-n-1} y^n \right) = \sum_{n=0}^{\infty} (-1)^n x^{n(n+1)/2} y^{n(n-1)/2}.
$$

This when interpreted combinatorially yields the following

**Theorem I.** The excess of the number of partitions of $(n, m)$ into even number of distinct parts of the type $(a, a - 1), (b - 1, b), (c, c)$ over those into odd number of such parts is $(-1)^r$ or 0 according as $(n, m)$ is of the type $(r(r+1)/2, r(r-1)/2)$ or not.

Let $a(n, m)$ denote the number of partitions of $(n, m)$ into distinct parts $(a, a - 1), (b - 1, b)$ so that we have the generating function

$$
\sum_{n,m=0}^{\infty} a(n, m) x^n y^m = \prod_{n=1}^{\infty} \left( 1 + x^n y^{n-1} \right) \left( 1 + x^{-n-1} y^n \right).
$$

In 1.1 making the substitution $q^2 = xy, t^2 = x/y$ we obtain

$$
\prod_{n=1}^{\infty} \left( 1 + x^n y^{n-1} \right) \left( 1 + x^{-n-1} y^n \right) = \left\{ \prod_{n=1}^{\infty} \left( 1 - x^n y^n \right) \right\}^{-1} \left\{ \sum_{n=0}^{\infty} x^{n(n+1)/2} y^{n(n-1)/2} \right\}
$$

$$
= \left\{ \sum_{n=1}^{\infty} p(n) x^n y^n \right\} \left\{ \sum_{n=0}^{\infty} x^{n(n+1)/2} y^{n(n-1)/2} \right\}.
$$

Equating coefficients Carlitz [2] obtained

$$
a(n, m) = p(n - \frac{1}{2}(n - m)(n - m + 1)).
$$

Conversely if one can prove this result combinatorially it yields a proof of Jacobi’s Identity, such a proof has been obtained by Wright in a forthcoming paper by setting up a 1-1 correspondence between the two types of partitions.

This is done by placing a triangular array of $(n - m) (n - m + 1)/2$ dots on the graph of each partition of $n - \frac{1}{2}(n - m) (n - m + 1)$, the columns under the diagonal and rows on the right side determine uniquely parts $(a, a - 1), (b - 1, b)$ of $(n, m)$.

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If one can prove Theorem I by combinatorial arguments similar to Franklin’s proof of Euler identity

\[
\prod_{r=1}^{\infty} (1 - x^r) = \sum_{\infty} (-1)^a x^{(2a+1)/2},
\]

it will yield a combinatorial proof of Jacobi’s Identity. The method of proof will depend on setting up a 1-1 correspondence between the partitions into even number of distinct parts and odd number of distinct parts of the type \((a, a - 1), (b - 1, b), (c, c)\); such a correspondence has to be 1-1 both ways. By means of simple operations one can change the parity of the number of parts except in the case when \((n, m)\) is of the type \((r(r + 1)/2, r(r - 1)/2)\), the parity of whose partition \((r, r - 1), (r - 1, r - 2), \ldots, (2, 1), (1, 0)\) cannot be changed and thus the excess in this case is \((-1)^r\) and \(0\) in other cases.

Gordon [1] has generalized Jacobi’s Identity such that there are five products on the left side, i.e.,

\[
\prod_{n=1}^{\infty} (1 - q^n)(1 - q^{2n-1}t)(1 - q^{2n-1}t^{-1})(1 - q^{4n-4}t^2)(1 - q^{4n-4}t^{-2})
\]

\[
= \sum_{\infty} q^{3n^2-2n}(t^{3n} + t^{-3n} - t^{3n-2} - t^{-3n+2}).
\]

Again put \(q^2 = xy, t^3 = x/y\) to obtain

\[
\prod_{n=1}^{\infty} (1 - x^n y^n)(1 - x^{n-1} y^{n-1})(1 - x^{2n-1} y^{2n-3})(1 - x^{2n-3} y^{2n-1})(1 - x^{n-1} y^n)
\]

\[
= \sum_{\infty} x^{(3n^2+n)/2} y^{(3n^2-5n)/2} + x^{(3n^2-5n)/2} y^{(3n^2+n)/2} - x^{(3n^2+n-2)/2} y^{(3n^2-5n+2)/2} - x^{(3n^2-5n+2)/2} y^{(3n^2+n-2)/2}.
\]

Let \(C(c, m)\) denote number of partitions of \((n, m)\) into vectors of the type \((a, a), (b, b - 1), (c - 1, c), (2d - 1, 2d - 3), (2e - 3, 2e - 1)\); thus the generating function is given by

\[
\prod_{n=1}^{\infty} \left(1 - x^n y^n\right) \left(1 - x^{n-1} y^{n-1}\right) \left(1 - x^{2n-1} y^{2n-3}\right) \left(1 - x^{2n-3} y^{2n-1}\right) \left(1 - x^{n-1} y^n\right)^{-1}
\]

\[
= \sum_{n,m=0}^{\infty} C(n, m) x^n y^m.
\]

Thus (1.6) yields the recurrence relation

\[
\sum C\left(n - \frac{3r^2 + r}{2}, m - \frac{3r^2 - 5r}{2}\right) + \sum C\left(n - \frac{3r^2 - 5r}{2}, m - \frac{3r^2 + r}{2}\right)
\]

\[
- \sum C\left(n - \frac{3r^2 + r - 2}{2}, m - \frac{3r^2 + 5r + 2}{2}\right)
\]

\[
- \sum C\left(n - \frac{3r^2 - 5r + 2}{2}, m - \frac{3r^2 - r - 2}{2}\right) = 0.
\]

Change \(x\) to \(-x\), \(y\) to \(-y\) in (1.5) to obtain
\[
\prod_{n=1}^{\infty} \left( 1 - x^n y^n \right) \left( 1 + x^{n-1} y^n \right) \left( 1 - x^{2n-1} y^{2n-3} \right) \left( 1 - x^{2n-3} y^{2n-1} \right)
\]

(1.8) 

\[
= \sum_{n=0}^{\infty} (-1)^n x \left( \frac{3n^2+n}{2} y \left( \frac{3n^2-5n}{2} \right) + \sum_{n=0}^{\infty} (-1)^n \left( \frac{3n^2-5n}{2} y \left( \frac{3n^2+n}{2} \right) \right)
\]

If \( D(n, m) \) denotes the number of partitions of \((n, m)\) into parts of the type \((2d-1, 2d-3)\), \((2e-3, 2e-1)\). We obtain a relation between \(\alpha(n, m)\) and \(D(n, m)\) by writing (1.8) in the form

\[
\sum (-1)^\lambda (xy)^{\lambda(3\lambda+1)/2} \left\{ \sum_{n,m=0}^{\infty} \alpha(n, m) x^n y^m \right\}
\]

(1.9) 

\[
= \left\{ \sum_{n,m=0}^{\infty} D(n, m) x^n y^m \right\} \left\{ \sum (-1)^n \left( \frac{3n^2+n}{2} y \left( \frac{3n^2-5n}{2} \right) \right)
\]

\[
+ (-1)^n \left( \frac{3n^2-5n}{2} y \left( \frac{3n^2+n}{2} \right) \right) + (-1)^{n+1} \left( \frac{3n^2-5n+2}{2} y \left( \frac{3n^2-n+2}{2} \right) \right)
\]

\[
+ (-1)^{n+1} \left( \frac{3n^2-5n+2}{2} y \left( \frac{3n^2-n+2}{2} \right) \right)
\]

and equating coefficients.

1.5 can also be written as

\[
\sum_{n=0}^{\infty} (-1)^n x_n (n+1) y_{n-1} = \left\{ \sum_{n,m=0}^{\infty} D(n, m) x^n y^m \right\}
\]

(1.10) 

\[
\left\{ \sum x \left( \frac{3n^2+n}{2} y \left( \frac{3n^2-5n}{2} \right) \right) + \sum x \left( \frac{3n^2+5n}{2} y \left( \frac{3n^2+n}{2} \right) \right)
\]

\[
- \sum_{n=0}^{\infty} x \left( \frac{3n^2-n-2}{2} y \left( \frac{3n^2-5n+2}{2} \right) - \sum x \left( \frac{3n^2-5n+2}{2} y \left( \frac{3n^2+n+2}{2} \right) \right)
\]

\]

equating coefficients

\[
\sum D \left( n - \frac{3r^2 + r}{2}, m - \frac{3r^2 - 5r}{2} \right)
\]

(1.11) 

\[
+ \sum D \left( n - \frac{3r^2 - 5r}{2}, m - \frac{3r^2 + r}{2} \right)
\]

\[
- \sum D \left( n - \frac{3r^2 - 5r + 2}{2}, m - \frac{3r^2 + r - 2}{2} \right)
\]

\[
- \sum D \left( n - \frac{3r^2 + r - 2}{2}, m - \frac{3r^2 - 5r + 2}{2} \right) = (-1)^r \text{ or 0}
\]

according as \((n, m)\) is of the type \((r(r+1)/2, r(r-1)/2)\) or not. The Jacobi Identity

\[
\prod_{n=1}^{\infty} \left( 1 - x^n y^n \right) \left( 1 + x^{n-1} y^n \right) \left( 1 - x^n y^{n-1} \right) = \sum_{n,m=0}^{\infty} \alpha(n, m) x^n y^m
\]

(1.12) 

\[
= \sum_{n=0}^{\infty} x \left( \frac{(n+1)/2, n(n-1)/2} \right)
\]

can be written as

\[
\left\{ \sum (-1)^\lambda (xy)^{\lambda(3\lambda+1)/2} \left\{ \sum_{n,m=0}^{\infty} \alpha(n, m) x^n y^m \right\}ight\} x^{r+1/2} y^{r-1/2}
\]

(1.13)
Thus equating coefficients
\[
\sum_{\lambda} (-1)^{\lambda} \left( n - \lambda \left( \frac{3\lambda \pm 1}{2} \right), m - \lambda \left( \frac{3\lambda \pm 1}{2} \right) \right) = 1 \text{ or } 0
\]
according as \((n, m)\) is or is not of the type \((r(r + 1)/2, r(r - 1)/2)\).

In the case of the number of partitions of an integer we have the well-known result that the number of partitions of \(n\) into odd parts is equal to the number of partitions of \(n\) into distinct parts. We can prove the following generalization of this result for vector partitions.

**Theorem II.** The number of partitions of \((n_1, n_2, \ldots, n_s)\) into vectors with at least one component odd is equal to the number of partitions of \((n_1, n_2, \ldots, n_s)\) into distinct parts (vectors). Note, the same result holds if the parts are required to have non-zero components.

**Proof.** Denote the generating function of unrestricted vector partitions by
\[
f(x_1, x_2, \cdots, x_s) = \prod_{k_i \geq 0} (1 - x_1^{k_1} x_2^{k_2} \cdots x_s^{k_s})^{-1}
\]
and notice that the generating function for the number of partitions with at least one component odd is
\[
g(x_1, \cdots, x_s) \prod_{j_i \geq 0} \{(1 - x_1^{j_1} x_2^{j_2} \cdots x_s^{j_s})\}^{-1}
\]
where at least one \(j_i\) is odd.

This is connected with \(f(x_1, \cdots, x_s)\) by
\[
g(x_1, \cdots, x_s) = f(x_1, \cdots, x_s) \prod_{k_i \geq 0} (1 + x_1^{k_1} x_2^{k_2} \cdots x_s^{k_s})
\]
and this proves the result.

Let
\[
f(x) = \left\{ \prod_{n=1}^{\infty} (1 - x^n) \right\}^{-1},
\]
\[
g(x) = \frac{f(x)}{f(x^2)} = \sum_{n=0}^{\infty} x^{n(n+1)/2},
\]
\[
\theta(x) = \sum_{n=0}^{\infty} x^{n^2}.
\]

Gordon [1] has shown that
\[
F(x) = \frac{f(x^2)f(x^3)}{f(x^2)f(x^3)} = g(x) - 3xg(x^3),
\]
\[
G(x) = \frac{f(x^3)f(x^3)f(x^{15})}{f(x)f(x^4)f(x^8)^2} = \frac{3}{2} \theta(x^3) - \frac{1}{2} \theta(x).
\]
Thus
\[ \int_0^1 x^{s^2-1} G(x) \, dx = \frac{3}{2} \sum_{n=1}^{\infty} \frac{1}{9n^2 + s^2} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2 + s^2} \]
\[ = \frac{\pi}{2s} \coth \left( \frac{\pi s}{3} \right) - \frac{\pi}{2s} \coth \left( \frac{\pi s}{3} \right) \]
when \( s^2 \to 0 \)

Also
\[ \int_0^1 x^{s^2} \{F(x) - 1\} \, dx = \sum_{n=1}^{\infty} \frac{1}{n^2 + n + s^2} - 3 \sum_{n=1}^{\infty} \frac{1}{9n^2 + 9n + 2} + s^2 \]
when \( s^2 \to 0 \).

We obtain
\[ \int_0^1 \frac{x}{x} \{F(x) - 1\} \, dx = 2 \sum_{n=1}^{\infty} \frac{1}{n^2 + n} - 6 \sum_{n=1}^{\infty} \frac{1}{9n^2 + 9n + 2} \]
but \( \sum_{n=1}^{\infty} 1/(n^2 + n) = 1 \). Thus
\[ \int_0^1 \frac{x}{x} \{F(x) - 1\} \, dx = 2 - 6 \sum_{n=1}^{\infty} \frac{1}{9n^2 + 9n + 2} \]

The author recently extended the table of values \( q_r(n, m) \) to \( n, m = 1(1) 49, r = 1(1) 98 \). These tables display the unique maxima property of \( P_r(n, m) \) the number of partitions of \( (n, m) \) into exactly \( r \) parts with positive components. Szekeres [3] proved this result for \( P_r(n) \) the number of partitions of \( n \) into exactly \( r \) parts. The value of \( r = r_0 \) for which such a maxima occurs was also obtained by Szekeres. It seems reasonable to conjecture that \( P_r(n_1, n_2, \cdots, n_k) \) attains a unique maxima for fixed \( n_i \) and \( r \) varying, to locate the position of the maxima is still another problem, we hope these numerical results will be useful in establishing these results.

Here we list the values of \( q_r(n, m) \) and \( P_r(n, m) \). The number of partitions of \( (n, m) \) into a most \( r \) parts and into exactly \( r \) parts with positive components respectively for \( n = m = 49, r = 1(1) 98 \). These calculations were performed on IBM7072 at the University of Arizona Computing Centre.

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