is compared with $E_n(x)$, $E_{n-1}(x)$, and $E_n(x) + e_n(x)$. Even at $x^2 = 1$ the improved approximation has only about one per cent error compared to forty per cent for $E_n(x)$.

### Accuracy of Asymptotic Approximations

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Bell Telephone Laboratories, Incorporated
Whippany, New Jersey

**A One-Step Method for the Numerical Solution of Second Order Linear Ordinary Differential Equations**

By J. T. Day

In this paper we shall give a one-step method for the numerical solution of second order linear ordinary differential equations based on Hermitian interpolation and the Lobatto four-point quadrature formula. One-step methods based on quadrature were introduced into the literature by Hammer and Hollingsworth [3]; for subsequent work see Morrison and Stoller [7], and Henrici [5].

Throughout our discussion we shall assume that the functions $N(x), f(x), g(x)$ of the differential equation $y'' = N(x)y' + f(x)y + g(x)$ are sufficiently differentiable to ensure that the derivations we give are valid in any context in which they are used.

In order to simplify somewhat the discussion of the method under consideration we shall first treat the differential equation $y'' = f(x)y' + g(x), y(x_0) = y_0, y'(x_0) = y'_0$. The necessary modifications for the general second order differential equation $y'' = N(x)y' + f(x)y + g(x)$ will be given later.

After integrating the above differential equation from $x_0$ to $x_1 = x_0 + h \ (h > 0)$, we obtain the system of integral equations:

$$y'(x_0 + h) = y'(x_0) + \int_{x_0}^{x_0 + h} [f(\tau)y(\tau) + g(\tau)] d\tau,$$

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We shall approximate the above integrals by the Lobatto four-point quadrature formulae on the interval \([x_0, x_0 + h]\), cf. [6],

\[
\int_{x_0}^{x_0 + h} F(x) \, dx = \frac{h}{2} \sum_{k=1}^{4} W_k \, F(\tau_k) + R_4.
\]

Here \(W_1 = W_4 = \frac{1}{3}, W_2 = W_3 = \frac{4}{3}\).

\[\tau_1 = x_0, \quad \tau_2 = x_0 + (5 - \sqrt{5})h/10, \quad \tau_3 = x_0 + (5 + \sqrt{5})h/10,\]

\[\tau_4 = x_0 + h, \quad R_4 = -\frac{4h^3P^{11}(\xi)}{3 \cdot 2^5 \cdot 15750},\]

where \(x_0 < \xi < x_0 + h\).

In order to shorten the succeeding calculations we denote \((5 - \sqrt{5})/10\) by \(r\), \((5 + \sqrt{5})/10\) by \(s\).

We have, approximating the integrals of (1) and (2) by the above quadrature formula,

\[
y'(x_0 + h) = y'(x_0) + \frac{h}{2} \sum_{k=1}^{4} W_k [f(\tau_k)y(\tau_k) + g(\tau_k)] + T_0,
\]

\[
y(x_0 + h) = y(x_0) + h y'(x_0)
\]

\[
+ \frac{h}{2} \sum_{k=1}^{4} W_k (x_0 + h - \tau_k)[f(\tau_k)y(\tau_k) + g(\tau_k)] + \tilde{T}_0.
\]

\((T_0\text{ and } \tilde{T}_0\text{ will be discussed in detail later.})\)

We must know \(y(\tau_2), y'(\tau_2)\) in order to apply the above formulae as a numerical method. We do this as follows. In addition to \(y(x_0), y'(x_0), y''(x_0)\), we suppose we know \(y(x_0 + h), y'(x_0 + h), y''(x_0 + h)\); we fit this data to a Hermite interpolating polynomial, cf. [6]:

\[
y(x_0 + th) = y(x_0)[1 - t^3 + 3t^2(t - 1) - 6t^3(t - 1)^2]
+ y'(x_0)[t - t^3 + 2t^2(t - 1) - 3t^3(t - 1)^2]h
+ y''(x_0)[t^2 - 2t^3 + t^4 - t^5(t - 1)^2]h^2/2
\]

\[
+ y'(x_1)[t^3 - 3t^2(t - 1) + 6t^3(t - 1)^2]
+ y''(x_1)[t^2(t - 1) - 3t^3(t - 1)^2]h
+ y''(x_1)[t^3(t - 1)^2]h^2/2 + y^{VI}(\xi_1) t^6(t - 1)^3h^6/720,
\]

where \(x_0 < \xi_1 < x_0 + h, 0 \leq t \leq 1\).

Using the differential equation and the abbreviated form for \(y(x_0 + th)\),

\[
y(x_0 + th) = A(t)y(x_0) + B(t)y'(x_0)h + C(t)y''(x_0)h^2/2
+ D(t)y(x_1) + E(t)y'(x_1)h + F(t)y''(x_1)h^2/2 + H(t),
\]

\((T_0\text{ and } \tilde{T}_0\text{ will be discussed in detail later.})\)

We must know \(y(\tau_2), y'(\tau_2)\) in order to apply the above formulae as a numerical method. We do this as follows. In addition to \(y(x_0), y'(x_0), y''(x_0)\), we suppose we know \(y(x_0 + h), y'(x_0 + h), y''(x_0 + h)\); we fit this data to a Hermite interpolating polynomial, cf. [6]:

\[
y(x_0 + th) = y(x_0)[1 - t^3 + 3t^2(t - 1) - 6t^3(t - 1)^2]
+ y'(x_0)[t - t^3 + 2t^2(t - 1) - 3t^3(t - 1)^2]h
+ y''(x_0)[t^2 - 2t^3 + t^4 - t^5(t - 1)^2]h^2/2
\]

\[
+ y'(x_1)[t^3 - 3t^2(t - 1) + 6t^3(t - 1)^2]
+ y''(x_1)[t^2(t - 1) - 3t^3(t - 1)^2]h
+ y''(x_1)[t^3(t - 1)^2]h^2/2 + y^{VI}(\xi_1) t^6(t - 1)^3h^6/720,
\]

where \(x_0 < \xi_1 < x_0 + h, 0 \leq t \leq 1\).

Using the differential equation and the abbreviated form for \(y(x_0 + th)\),

\[
y(x_0 + th) = A(t)y(x_0) + B(t)y'(x_0)h + C(t)y''(x_0)h^2/2
+ D(t)y(x_1) + E(t)y'(x_1)h + F(t)y''(x_1)h^2/2 + H(t),
\]
we obtain
\[ y(x_0 + th) = y(x_0)[A(t) + f(x_0)C(t)h^2/2] + y'(x_0)B(t)h + y(x_1)[A(t) + f(x_0)C(t)h^2/2] \]
\[ + y'(x_1)E(t)h + [g(x_0)C(t) + g(x_1)D(t)]h^2/2 + H(t). \]

Letting
\[ \alpha(t, h) = A(t) + f(x_0)C(t)h^2/2, \]
\[ \gamma(t, h) = D(t) + F(t)f(x_1)h^2/2, \]

and \( A(r) = A, B(r) = B, \) etc., \( f(x_0) = f_0, f(x_0 + rh) = f_r, \) etc., we have, substituting Eq. (8) into Eqs. (4) and (5) for the values of \( y(\tau_3), y'(\tau_3), \) two linear equations to be solved for \( y(x_0 + h), y'(x_0 + h). \) They reduce to
\[ \begin{pmatrix} y(x_0 + h) \\ y'(x_0 + h) \end{pmatrix} = \begin{pmatrix} A_0 B_0 \\ y(x_0) \end{pmatrix} + A_0 G_0 + A_0 T_0^*, \]
where \( A_0 = \gamma / \det(C), \) in which
\[ C = \begin{bmatrix} 1 - \frac{5h^2}{12} (f_r E_r + f_s E_s) & \frac{5h^2}{12} (s f_r E_r + r f_s E_s) \\ \frac{5h}{12} [f_r \gamma_r + f_s \gamma_s] + \frac{h f_1}{12} & 1 - \frac{5h^2}{12} (s f_r \gamma_r + r f_s \gamma_r) \end{bmatrix}. \]

It is easily seen that \( \det(C) \neq 0 \) if \( h \) is sufficiently small. \( B_0 \) denotes the matrix
\[ B_0 = \begin{bmatrix} 1 + \frac{h^2}{12} f_o + \frac{5h^2}{12} (s f_r \alpha_r + r f_s \alpha_s) & h + \frac{5h^2}{12} (s f_r B_r + r f_s B_s) \\ \frac{h}{12} f_o + \frac{5h}{12} (f_r \alpha_r + f_s \alpha_s) & 1 + \frac{5h^2}{12} (f_r B_r + f_s B_s) \end{bmatrix}, \]
\( G_0 \) denotes the column vector
\[ G_0 = \begin{bmatrix} \frac{h^2}{12} g_o + \frac{5h^2}{12} (s g_r + r g_s) + \frac{5h^4}{24} [g_o (s C_r f_r + r C_s f_s) + g_1 (s F_r f_r + r F_s f_s)] \\ \frac{h}{12} [g_o + g_1] + \frac{5h^2}{12} [g_r + g_s] + \frac{5h^4}{24} [g_1 (C_r f_r + C_s f_s) + g_2 (F_r f_r + F_s f_s)] \end{bmatrix}. \]

In order to obtain an upper bound for the truncation error vector we consider the quantities \( H(t), T_0, \bar{T}_0. \)

By the definition of \( H(t) \) given above,
\[ H(r) = -h^6 r^3 y^{VI}(\xi_r)/720 = -h^6 y^{VI}(\xi_r)/90,000. \]
\[ H(s) = -h^6 y^{VI}(\xi_s)/90,000 \]
where \( \xi_r \) and \( \xi_s \) are in the open interval \( (x_0, x_0 + h). \)

\( T_0 \) and \( \bar{T}_0 \) are given by the following formulae:
\[ T_0 = -[f(x) y(x) + g(x)]^{VI}_{t_1} h^7/(96 \cdot 15750), \]
\[ \bar{T}_0 = -[(x_0 + h - \tau)(f(x) y(x) + g(x))]^{VI}_{t_1} h^7/(96 \cdot 15750), \]
where \( \xi_3 \) and \( \xi_4 \) are in the open interval \( (x_0, x_0 + h). \)
Thus we obtain for $T_0^*$ the following:

\[
T_0^* = \left( \frac{5h^2}{12} \left[ g_{fr} H_r + r_{fr} H_i \right] + T_0 \right)
\]

Thus the approximate solution at $x_i$ given by

\[
\begin{pmatrix} y_i \\ y_i' \end{pmatrix} = \tilde{A}_0 B_0 \begin{pmatrix} y_0 \\ y_0' \end{pmatrix} + \tilde{A}_0 G_0
\]

has local truncation error $O(h^7)$.

We shall consider three computational examples. We have written programs for the CDC 1604 computer, FORTRAN (single-precision), for the following methods: Runge-Kutta, Numerov \cite{4} and Gautschi (Stromer interpolation of trigonometric order two) \cite{2}. We have used the same estimate of the period $T = \pi/5$ as Gautschi's article \cite{2}.

Example 1. A Mathieu differential equation $y'' + 100(1 - \cos(2x))y = 0$, with the initial conditions taken as $x = 0$, $y(0) = 1$, $y'(0) = 0$. After starting Numerov's and Gautschi's methods by the Runge-Kutta method we obtain the values shown in Table 1.

Example 2. Bessel differential equation $y'' + \left( 100 + \frac{1}{4}x^2 \right)y = 0$. We take the initial conditions at $x = 1$, such that the solution is $\sqrt{x} J_0(10x)$. We have $h$ as 0.02 again. We have taken the initial values from \cite{1} to 10D. For Numerov's and Gautschi's method we have taken the other starting values from the table also (Table 2).

Example 3. Our last example is the differential equation $y'' = (1 + x^2)y$. The initial conditions in this example were chosen at $x = 0$ so that the solution is $e^{x^2}$. We again take $h = 0.02$ and obtain the results of Table 3, after taking all the necessary starting values as exact.

The general second order equation $y'' = N(x)y' + f(x)y + g(x)$ can be treated by the above techniques, if one treats the $N(x)y'$ term by integration by parts. One may also use a well-known transformation \cite{6} to eliminate the $y'$ term from the above differential equation. The procedure one should use depends primarily on whether or not $N(x)$ is explicitly integrable.

It is well known that two-point boundary value problems of the form $y'' = f(x)y + g(x)$, $y(a) = A$, $y(b) = B$, $-\infty < a < b < \infty$, $f(x) > 0$ can be solved by initial value techniques either by the method of superposition or by the numerical construction of the Green's function of the above differential equation. We have made calculations on problems of the above type with the one-step method under consideration and have found the results to be quite satisfactory.

I am especially indebted to Professor P. C. Hammer for many discussions on the numerical solution of differential equations, and to the Wisconsin Alumni Research Foundation and the National Science Foundation who, through the Graduate Research Committee, made available to me the computing facilities of the Numerical Analysis Laboratory of the University of Wisconsin.

Numerical Analysis Laboratory
University of Wisconsin
Madison, Wisconsin
### Table 1

**Mathieu Differential Equation**

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**Bessel Differential Equation**

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### Table 3

**Differential Equation** $y'' = (1 + x^2)y$

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