and (ii) in the case of rhombic primitive period-parallelogram

$$\sigma_{2s}(c') = (-1)^s(2c)^{2s}\sigma_{2s}(c) \qquad (cc' = \tfrac{1}{4}). \tag{6}$$

The computation has been carried out up to $2s = 50$ with adequate guarding figures provided for σ_4 and σ_6. The values are then rounded off to 16D. Individual check is made on the last two coefficients by direct summation of the double series. The results up to $2s = 20$ are shown in Tables 1 and 2. In Table 2, the values of σ_4 and σ_6 are not included, which may be found in reference 2. The complete table is deposited in the UMT file in the office of the journal.

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A Method for the Computation of the Error Function of a Complex Variable

By Otto Neall Strand

Abstract. This paper presents a method of computing $\operatorname{erf} z \equiv (2/\sqrt{\pi}) \int_0^z e^{-u^2}\, du$, where z is complex. It is shown that $\operatorname{erfc} z \equiv 1 - \operatorname{erf} z$ has no zeros in the right-hand half plane. An estimate of $|\operatorname{erfc} z|$ is derived.

The error function of a complex variable, denoted by erf z, is defined by the equation $\operatorname{erf} z = (2/\sqrt{\pi})\int_0^z e^{-u^2}\, du$, where z is complex. This function arises in many problems of physics and engineering. Several methods [1], [2], [3] have been devised for the computation of erf z and closely-related functions, and several tabulations [4], [5], [6] have been made. The method to be described below has two features which make it relatively simple to use: (1) the phase enters in a simple explicit manner; and (2) the major portion of the computation consists of the accumulation of two series of positive terms for which each term (after the first) may be calculated by a simple recursion without the use of transcendental functions. For the particular FORTRAN double-precision programs which were written for comparison, the average computing time for the method of this paper was found to be approximately $\frac{7}{10}$ of that for Salzer's first method [7] for an equally-spaced grid of points throughout the region defined by $0 < |z| < 6.6$ and $0 \leqq \arg z < \pi/2$. The relative difference between results from the two methods was less than 10^{-13} throughout this region.

Since the relations $\operatorname{erf}(-z_0) = -\operatorname{erf} z_0$ and $\overline{\operatorname{erf}(z_0)} = \operatorname{erf}(\bar{z}_0)$ may always be employed to reduce the computation to one involving z_0 in the first quadrant, the

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following derivation is restricted to the computation of erf z_0, where $z_0 = x_0 + iy_0$, $x_0 > 0$ and $y_0 \geqq 0$. The case $x_0 = 0$ is not covered by this method.

By Cauchy's theorem:

$$\text{(1)} \qquad \operatorname{erfc} z_0 \equiv 1 - \operatorname{erf} z_0 = \frac{2}{\sqrt{\pi}} \int_C e^{-u^2}\, du,$$

where C is the hyperbola $xy = x_0 y_0 = v_0$ for which the integrand has constant phase, described in the direction of increasing x from $x = x_0$ to $x = \infty$. Reduction of the line integral to definite integrals gives the result

$$\text{(2)} \quad \operatorname{erfc} z_0 = H_1 \cos 2v_0 - y_0 H_2 \sin 2v_0 + i[-H_1 \sin 2v_0 - y_0 H_2 \cos 2v_0],$$

where

$$\text{(3)} \qquad \begin{aligned} H_1 &= \frac{2}{\sqrt{\pi}} \int_{x_0}^{\infty} \exp\left[-\left(x^2 - \frac{v_0^2}{x^2}\right)\right] dx, \\ H_2 &= \frac{2x_0}{\sqrt{\pi}} \int_{x_0}^{\infty} \frac{1}{x^2} \exp\left[-\left(x^2 - \frac{v_0^2}{x^2}\right)\right] dx. \end{aligned}$$

We expand the integrands of H_1 and H_2 in series as follows:

$$\text{(4)} \qquad \begin{aligned} H_1 &= \frac{2}{\sqrt{\pi}} \int_{x_0}^{\infty} e^{-x^2} \left(\sum_{n=0}^{\infty} \frac{(v_0)^{2n}}{n!\, x^{2n}}\right) dx, \\ H_2 &= \frac{2x_0}{\sqrt{\pi}} \int_{x_0}^{\infty} e^{-x^2} \left(\sum_{n=0}^{\infty} \frac{(v_0)^{2n}}{n!\, x^{2n+2}}\right) dx. \end{aligned}$$

Since all terms in the series are positive, term-wise integration can be justified by the Lebesgue Monotone Convergence Theorem [8], so that

$$\text{(5)} \qquad \begin{aligned} H_1 &= \sum_{n=0}^{\infty} \gamma_n v_0^{2n}, \\ H_2 &= x_0 \sum_{n=0}^{\infty} (n+1) \gamma_{n+1} v_0^{2n}, \end{aligned}$$

where

$$\text{(6)} \qquad \gamma_n = \frac{2}{n!\,\sqrt{\pi}} \int_{x_0}^{\infty} \frac{e^{-x^2}}{x^{2n}}\, dx, \qquad n = 0, 1, 2, \cdots.$$

Since $\gamma_0 = \operatorname{erfc} x_0$, it can be obtained from existing methods. To obtain the other γ's we integrate γ_{n+1} by parts to obtain

$$\text{(7)} \quad \gamma_{n+1} = \frac{2}{(2n+1)\sqrt{\pi}} \left[\frac{e^{-x_0^2}}{(n+1)!\, x_0^{2n+1}} - \frac{\sqrt{\pi}}{n+1} \gamma_n\right], \qquad n = 0, 1, 2, \cdots.$$

The method of computation consists of computing the series (5), where the coefficients are obtained recursively by (7). The values of H_1 and H_2 are then substituted into (2) to obtain erfc z_0, from which erf z_0 is obtainable by (1).

Although the following results are of some interest, they do not pertain directly to the method of computation. By (2),

$$\text{(8)} \qquad |\operatorname{erfc} z_0| = \sqrt{(H_1^2 + y_0^2 H_2^2)}.$$

Therefore erfc z has no zeros in the right-hand half plane. This property is evident in examining the contour charts due to Laible [9]. It can be shown [10] that

$$\int_{x_0}^{\infty} e^{-x^2}\,dx < \frac{e^{-x_0^2}}{2x_0} \quad \text{for } x_0 > 0.$$

Therefore

$$H_1 < \frac{\exp(y_0^2 - x_0^2)}{x_0\sqrt{\pi}}, \qquad H_2 < \frac{\exp(y_0^2 - x_0^2)}{x_0^2\sqrt{\pi}}. \tag{9}$$

Combination of (8) with (9) gives the following estimate for the absolute deviation of erf z_0 from 1:

$$|\operatorname{erfc} z_0| < \frac{e^{y_0^2 - x_0^2}}{x_0\sqrt{\pi}} \sqrt{(1 + y_0^2/x_0^2)}. \tag{10}$$

This estimate may be useful in some cases to determine if erf z_0 may be approximated by 1 with sufficient accuracy.

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