and (ii) in the case of rhombic primitive period-parallelogram

\[ \sigma_{2s}(c') = (-1)^s (2c)^2 \sigma_{2s}(c) \quad (cc' = \frac{1}{2}). \]

The computation has been carried out up to \(2s = 50\) with adequate guarding figures provided for \(\sigma_4\) and \(\sigma_6\). The values are then rounded off to 16D. Individual check is made on the last two coefficients by direct summation of the double series. The results up to \(2s = 20\) are shown in Tables 1 and 2. In Table 2, the values of \(\sigma_4\) and \(\sigma_6\) are not included, which may be found in reference 2. The complete table is deposited in the UMT file in the office of the journal.

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**A Method for the Computation of the Error Function of a Complex Variable**

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Abstract. This paper presents a method of computing \(\text{erf } z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2} \, du\), where \(z\) is complex. It is shown that \(\text{erfc } z = 1 - \text{erf } z\) has no zeros in the right-hand half plane. An estimate of \(|\text{erfc } z|\) is derived.

The error function of a complex variable, denoted by \(\text{erf } z\), is defined by the equation \(\text{erf } z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2} \, du\), where \(z\) is complex. This function arises in many problems of physics and engineering. Several methods [1], [2], [3] have been devised for the computation of \(\text{erf } z\) and closely-related functions, and several tabulations [4], [5], [6] have been made. The method to be described below has two features which make it relatively simple to use: (1) the phase enters in a simple explicit manner; and (2) the major portion of the computation consists of the accumulation of two series of positive terms for which each term (after the first) may be calculated by a simple recursion without the use of transcendental functions. For the particular FORTRAN double-precision programs which were written for comparison, the average computing time for the method of this paper was found to be approximately 1.7 times that for Salzer's first method [7] for an equally-spaced grid of points throughout the region defined by \(0 < |z| < 6.6\) and \(0 \leq \arg z < \pi/2\). The relative difference between results from the two methods was less than \(10^{-13}\) throughout this region.

Since the relations \(\text{erf } (-z) = -\text{erf } z\) and \(\text{erf } (\bar{z}) = \text{erf } (\bar{z})\) may always be employed to reduce the computation to one involving \(z_0\) in the first quadrant, the

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following derivation is restricted to the computation of \( \text{erf} \, z_0 \), where \( z_0 = x_0 + i y_0 \), \( x_0 > 0 \) and \( y_0 \geq 0 \). The case \( x_0 = 0 \) is not covered by this method.

By Cauchy’s theorem:

(1) \[
\text{erfc} \, z_0 = 1 - \text{erf} \, z_0 = \frac{2}{\sqrt{\pi}} \int_C e^{-u^2} \, du,
\]

where \( C \) is the hyperbola \( xy = x_0 y_0 = v_0 \) for which the integrand has constant phase, described in the direction of increasing \( x \) from \( x = x_0 \) to \( x = \infty \). Reduction of the line integral to definite integrals gives the result

(2) \[
\text{erfc} \, z_0 = H_1 \cos 2v_0 - y_0 H_2 \sin 2v_0 + i[-H_1 \sin 2v_0 - y_0 H_2 \cos 2v_0],
\]

where

\[
H_1 = \frac{2}{\sqrt{\pi}} \int_{x_0}^{\infty} \exp \left[ -\left( x^2 - \frac{v_0^2}{x^2} \right) \right] \, dx,
\]

(3) \[
H_2 = \frac{2x_0}{\sqrt{\pi}} \int_{x_0}^{\infty} \frac{1}{x^2} \exp \left[ -\left( x^2 - \frac{v_0^2}{x^2} \right) \right] \, dx.
\]

We expand the integrands of \( H_1 \) and \( H_2 \) in series as follows:

\[
H_1 = \frac{2}{\sqrt{\pi}} \int_{x_0}^{\infty} e^{-x^2} \sum_{n=0}^{\infty} \frac{(v_0)^{2n}}{n! x^{2n}} \, dx,
\]

(4) \[
H_2 = \frac{2x_0}{\sqrt{\pi}} \int_{x_0}^{\infty} e^{-x^2} \sum_{n=0}^{\infty} \frac{(v_0)^{2n}}{n! x^{2n+2}} \, dx.
\]

Since all terms in the series are positive, term-wise integration can be justified by the Lebesgue Monotone Convergence Theorem [8], so that

\[
H_1 = \sum_{n=0}^{\infty} \gamma_n v_0^{2n},
\]

(5) \[
H_2 = x_0 \sum_{n=0}^{\infty} (n + 1) \gamma_{n+1} v_0^{2n},
\]

where

\[
\gamma_n = \frac{2}{n!} \int_{x_0}^{\infty} \frac{e^{-x^2}}{x^{2n}} \, dx, \quad n = 0, 1, 2, \ldots.
\]

Since \( \gamma_0 = \text{erfc} \, x_0 \), it can be obtained from existing methods. To obtain the other \( \gamma \)'s we integrate \( \gamma_{n+1} \) by parts to obtain

\[
\gamma_{n+1} = \frac{2}{(2n + 1) \sqrt{\pi}} \left[ \frac{e^{-x_0^2}}{(n + 1) x_0^{2n+1}} - \frac{\sqrt{\pi}}{n + 1} \gamma_n \right], \quad n = 0, 1, 2, \ldots.
\]

The method of computation consists of computing the series (5), where the coefficients are obtained recursively by (7). The values of \( H_1 \) and \( H_2 \) are then substituted into (2) to obtain \( \text{erfc} \, z_0 \), from which \( \text{erf} \, z_0 \) is obtainable by (1).

Although the following results are of some interest, they do not pertain directly to the method of computation. By (2),

\[
| \text{erfc} \, z_0 | = \sqrt{(H_1^2 + y_0^2 H_2^2)}.
\]
Therefore erfc $z$ has no zeros in the right-hand half plane. This property is evident in examining the contour charts due to Laible [9]. It can be shown [10] that

$$
\int_{x_0}^{\infty} e^{-x^2} \, dx < \frac{e^{-x_0^2}}{2x_0} \quad \text{for} \quad x_0 > 0.
$$

Therefore

$$
H_1 < \frac{\exp \left( \frac{y_0^2 - x_0^2}{x_0 \sqrt{\pi}} \right)}{x_0 \sqrt{\pi}},
$$

(9)

$$
H_2 < \frac{\exp \left( \frac{y_0^2 - x_0^2}{x_0^2 \sqrt{\pi}} \right)}{x_0 \sqrt{\pi}}.
$$

Combination of (8) with (9) gives the following estimate for the absolute deviation of erf $z_0$ from 1:

$$
| \text{erfc} \, z_0 | < \frac{e^{y_0^2 - x_0^2}}{x_0 \sqrt{\pi}} \sqrt{1 + \frac{y_0^2}{x_0^2}}.
$$

(10)

This estimate may be useful in some cases to determine if erf $z_0$ may be approximated by 1 with sufficient accuracy.

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