

and (ii) in the case of rhombic primitive period-parallellogram

$$(6) \quad \sigma_{2s}(c') = (-1)^s (2c)^{2s} \sigma_{2s}(c) \quad (cc' = \frac{1}{4}).$$

The computation has been carried out up to  $2s = 50$  with adequate guarding figures provided for  $\sigma_4$  and  $\sigma_6$ . The values are then rounded off to 16D. Individual check is made on the last two coefficients by direct summation of the double series. The results up to  $2s = 20$  are shown in Tables 1 and 2. In Table 2, the values of  $\sigma_4$  and  $\sigma_6$  are not included, which may be found in reference 2. The complete table is deposited in the UMT file in the office of the journal.

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## A Method for the Computation of the Error Function of a Complex Variable

By Otto Neall Strand

**Abstract.** This paper presents a method of computing  $\operatorname{erf} z \equiv (2/\sqrt{\pi}) \int_0^z e^{-u^2} du$ , where  $z$  is complex. It is shown that  $\operatorname{erfc} z \equiv 1 - \operatorname{erf} z$  has no zeros in the right-hand half plane. An estimate of  $|\operatorname{erfc} z|$  is derived.

The error function of a complex variable, denoted by  $\operatorname{erf} z$ , is defined by the equation  $\operatorname{erf} z = (2/\sqrt{\pi}) \int_0^z e^{-u^2} du$ , where  $z$  is complex. This function arises in many problems of physics and engineering. Several methods [1], [2], [3] have been devised for the computation of  $\operatorname{erf} z$  and closely-related functions, and several tabulations [4], [5], [6] have been made. The method to be described below has two features which make it relatively simple to use: (1) the phase enters in a simple explicit manner; and (2) the major portion of the computation consists of the accumulation of two series of positive terms for which each term (after the first) may be calculated by a simple recursion without the use of transcendental functions. For the particular FORTRAN double-precision programs which were written for comparison, the average computing time for the method of this paper was found to be approximately  $\frac{1}{10}$  of that for Salzer's first method [7] for an equally-spaced grid of points throughout the region defined by  $0 < |z| < 6.6$  and  $0 \leq \arg z < \pi/2$ . The relative difference between results from the two methods was less than  $10^{-13}$  throughout this region.

Since the relations  $\operatorname{erf}(-z_0) = -\operatorname{erf} z_0$  and  $\overline{\operatorname{erf}(z_0)} = \operatorname{erf}(\overline{z_0})$  may always be employed to reduce the computation to one involving  $z_0$  in the first quadrant, the

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following derivation is restricted to the computation of  $\operatorname{erf} z_0$ , where  $z_0 = x_0 + iy_0$ ,  $x_0 > 0$  and  $y_0 \geq 0$ . The case  $x_0 = 0$  is not covered by this method.

By Cauchy's theorem:

$$(1) \quad \operatorname{erfc} z_0 \equiv 1 - \operatorname{erf} z_0 = \frac{2}{\sqrt{\pi}} \int_C e^{-u^2} du,$$

where  $C$  is the hyperbola  $xy = x_0y_0 = v_0$  for which the integrand has constant phase, described in the direction of increasing  $x$  from  $x = x_0$  to  $x = \infty$ . Reduction of the line integral to definite integrals gives the result

$$(2) \quad \operatorname{erfc} z_0 = H_1 \cos 2v_0 - y_0 H_2 \sin 2v_0 + i[-H_1 \sin 2v_0 - y_0 H_2 \cos 2v_0],$$

where

$$(3) \quad \begin{aligned} H_1 &= \frac{2}{\sqrt{\pi}} \int_{x_0}^{\infty} \exp \left[ -\left( x^2 - \frac{v_0^2}{x^2} \right) \right] dx, \\ H_2 &= \frac{2x_0}{\sqrt{\pi}} \int_{x_0}^{\infty} \frac{1}{x^2} \exp \left[ -\left( x^2 - \frac{v_0^2}{x^2} \right) \right] dx. \end{aligned}$$

We expand the integrands of  $H_1$  and  $H_2$  in series as follows:

$$(4) \quad \begin{aligned} H_1 &= \frac{2}{\sqrt{\pi}} \int_{x_0}^{\infty} e^{-x^2} \left( \sum_{n=0}^{\infty} \frac{(v_0)^{2n}}{n! x^{2n}} \right) dx, \\ H_2 &= \frac{2x_0}{\sqrt{\pi}} \int_{x_0}^{\infty} e^{-x^2} \left( \sum_{n=0}^{\infty} \frac{(v_0)^{2n}}{n! x^{2n+2}} \right) dx. \end{aligned}$$

Since all terms in the series are positive, term-wise integration can be justified by the Lebesgue Monotone Convergence Theorem [8], so that

$$(5) \quad \begin{aligned} H_1 &= \sum_{n=0}^{\infty} \gamma_n v_0^{2n}, \\ H_2 &= x_0 \sum_{n=0}^{\infty} (n+1) \gamma_{n+1} v_0^{2n}, \end{aligned}$$

where

$$(6) \quad \gamma_n = \frac{2}{n! \sqrt{\pi}} \int_{x_0}^{\infty} \frac{e^{-x^2}}{x^{2n}} dx, \quad n = 0, 1, 2, \dots$$

Since  $\gamma_0 = \operatorname{erfc} x_0$ , it can be obtained from existing methods. To obtain the other  $\gamma$ 's we integrate  $\gamma_{n+1}$  by parts to obtain

$$(7) \quad \gamma_{n+1} = \frac{2}{(2n+1)\sqrt{\pi}} \left[ \frac{e^{-x_0^2}}{(n+1)! x_0^{2n+1}} - \frac{\sqrt{\pi}}{n+1} \gamma_n \right], \quad n = 0, 1, 2, \dots$$

The method of computation consists of computing the series (5), where the coefficients are obtained recursively by (7). The values of  $H_1$  and  $H_2$  are then substituted into (2) to obtain  $\operatorname{erfc} z_0$ , from which  $\operatorname{erf} z_0$  is obtainable by (1).

Although the following results are of some interest, they do not pertain directly to the method of computation. By (2),

$$(8) \quad |\operatorname{erfc} z_0| = \sqrt{(H_1^2 + y_0^2 H_2^2)}.$$

Therefore  $\operatorname{erfc} z$  has no zeros in the right-hand half plane. This property is evident in examining the contour charts due to Laible [9]. It can be shown [10] that

$$\int_{x_0}^{\infty} e^{-x^2} dx < \frac{e^{-x_0^2}}{2x_0} \quad \text{for } x_0 > 0.$$

Therefore

$$(9) \quad \begin{aligned} H_1 &< \frac{\exp(y_0^2 - x_0^2)}{x_0\sqrt{\pi}}, \\ H_2 &< \frac{\exp(y_0^2 - x_0^2)}{x_0^2\sqrt{\pi}}. \end{aligned}$$

Combination of (8) with (9) gives the following estimate for the absolute deviation of  $\operatorname{erf} z_0$  from 1:

$$(10) \quad |\operatorname{erfc} z_0| < \frac{e^{y_0^2 - x_0^2}}{x_0\sqrt{\pi}} \sqrt{(1 + y_0^2/x_0^2)}.$$

This estimate may be useful in some cases to determine if  $\operatorname{erf} z_0$  may be approximated by 1 with sufficient accuracy.

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