Evaluation of the Integral
\[ I_n(b) = \frac{2}{\pi} \int_0^\infty \left( \frac{\sin x}{x} \right)^n \cos(bx) \, dx \]

By R. G. Medhurst and J. H. Roberts

This integral appears in a number of diverse problems (see, e.g., [1], [2], and [6]). The present authors required it in connection with certain operations involving "white" noise. For example, the intermodulation distortion generated by taking the \( n \)th power of a narrow-band, high-frequency white noise is proportional to \( I_n(b) \): \( b = 0 \) corresponds to the distortion level at the mid-band frequency, \( b = 1 \) to the distortion level at the edges of the band, \( b = 2 \) to the distortion level at frequencies spaced by twice the semi-bandwidth from the center frequency, and so on.

In closed form, \( I_n(b) \) is given [3] by

\[
I_n(b) = \frac{n}{2^{n-1}} \sum_{0 \leq r < (b+n)/2} \frac{(-1)^r(b + n - 2r)^{n-1}}{r! (n - r)!}, \quad 0 \leq b < n
\]

(where \( r \) takes integral values),

\[ = 0, \quad n \leq b < \infty. \]

(The lower limit of \( r \) is not correct in [3].) The special case of this formula for \( b = 0 \) is given in [4], [5], and [7]. This expression for \( I_n(b) \) is useful for small \( n \), but becomes prohibitively cumbersome, even for \( b = 0 \), as \( n \) increases beyond the range covered in [2]. One is thus led to seek a limiting form for large \( n \).

We consider first the case \( b = 0 \). For the smaller \( n \)'s it is practicable to express \( I_n(0) \) in rational form, and this is done up to \( n = 12 \) in [4].

The following Table 1 extends Grimsey's table up to \( n = 16 \). In [2], a ten-place table is given for \( 1 \leq n \leq 30 \). The entry for \( n = 30 \) is in error, and should read 0.2510485150.

Goddard [5] takes to task the author of [1] for his lack of rigour, and attempts to supply a deficiency in that reference by deriving what he claims to be an asymptotic expansion for \( I_n(0) \), applicable for large \( n \). In fact, while Goddard's method is of considerable interest it involves steps of which the region of applicability needs careful scrutiny. (He has, in addition, made an analytical error resulting in an incorrect coefficient for his final term.)

Goddard's approach is based on a relation which, after correcting some misprints in the version in [5], we can give as

\[
\log \left( \frac{\sin x}{x} \right) = -\sum_{K=1}^{\infty} \frac{B_K}{2K(2K)!} (2x)^{2K},
\]

where the \( B_K \)'s are the Bernoulli numbers.

Need for caution is immediately apparent, since the left-hand side becomes imaginary for \( x \) in the ranges \((2m - 1)\pi \) to \( 2m\pi \) \((m = 1, 2, 3, \ldots)\), while the right-hand side is always a sum of real terms. It can, in fact, be readily shown that the right-hand side diverges for \( x \geq \pi \). Goddard then writes

\[
\left( \frac{\sin x}{x} \right)^n = \exp \left[ -n \sum_{K=1}^{\infty} \frac{B_K}{2K(2K)!} (2x)^{2K} \right]
\]

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where the \( a_n \)’s are functions of \( n \) to be determined, and proceeds to integrate expression (1.1) term by term, using the relation

\[
\frac{2}{\pi} \int_{0}^{\infty} \exp \left( -\frac{nx^2}{6} \right) x^{2k} \, dx = \left( \frac{1}{2} \right)^{2k-1} \frac{(2K-1)!}{(K-1)!} \frac{6^k}{n^K} \sqrt{\frac{6}{\pi n}}.
\]

Table 1

<table>
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<th>( n )</th>
<th>( I_n(0) )</th>
</tr>
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<tr>
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The coefficients \( a_m \) are obtained by expanding the exponential expression (1) after removing the term \( \exp \left( -\frac{nx^2}{6} \right) \), and collecting powers of \( x \). Each term obtained by integrating the terms of (1.1) yields sums of inverse powers of \( n \) (together with the factor \( \sqrt{(6/\pi n)} \)), and these can be collected to yield a power series in \( 1/n \). Fortunately, there are only a finite number of contributions to each power of \( 1/n \).

Goddard obtained an expansion, in this way, up to \((1/n)^3\). However, the last term he gives is in error, since he failed to take account of all contributions. We have evaluated in closed form the coefficients of terms up to \((1/n)^6\). The labour increases rapidly with the order of the term, and after this point it only seemed worth computing the coefficients to a few significant figures. As far as it has been taken, the expansion is as follows:

\[
I_n(0) \approx \sqrt{\frac{6}{\pi n}} \left[ 1 - \frac{3}{20} \frac{1}{n} - \frac{13}{1120} \frac{1}{n^2} + \frac{27}{3200} \frac{1}{n^3} + \frac{52791}{3942400} \frac{1}{n^4} + \frac{482427}{66560000} \frac{1}{n^5} - \frac{124996631}{10035200000} \frac{1}{n^6} - 0.0386166 \frac{1}{n^7} - 0.027677 \frac{1}{n^8} + 0.12245 \frac{1}{n^9} + 0.4523 \frac{1}{n^{10}} \right].
\]

Presumably, this is a divergent series (though, since the coefficients are not known in analytical form, it is not easy to see how this could be proved), and Goddard claims it is asymptotic. In fact, after making the correction already mentioned to the last entry of the table of [2], formula (2) yields \( I_n(0) \) values in agreement to the full ten places with those tabulated over the range \( 17 \leq n \leq 30 \). At \( n = 16 \) there is a disagreement of one unit in the last place, while at \( n = 15 \) there is exact agreement. For \( n = 14, 13 \) and \( 12 \) the agreement is increasingly poor, the last two digits from formula (2) being 93, 56 and 02, respectively, whereas the corresponding exact values are 95, 45 and 52.

It should be emphasized that the discrepancy is not due to a failure of initial convergence. For \( n = 12 \), eleven terms of the series yield a sum +0.39392 55601 869, the final term being +0.00000 00000 029. This is to be compared with the true value
0.39392 55651 7: if the series were convergent, or were asymptotic in the sense that the error is less in magnitude than the last included term, it is clear that ten-place accuracy should have resulted.

The discrepancy has, in fact, a rather interesting source. The expression \((\sin x)/x)^n\), when plotted out, will clearly consist of a succession of peaks (positive for even \(n\), and of alternating sign for odd \(n\)) centered round \(x = 0, x = 4.4934\), etc. In the region \(0 \leq x \leq \pi\), containing the first peak, the series in expression (1.1) converges so that (1.1) represents the required expression with increasing accuracy as more terms are taken. But in the region containing the second peak, the exponential in (1.1) becomes, for moderate \(n\), vanishingly small to the order of accuracy in which we are interested, and the terms of the series are not such as to counterbalance the exponential. Consequently, (1.1) cannot include the contribution of the second and later peaks, and, in fact, it represents an approximation only to the contribution of the first peak to \(I_n(0)\).

This can readily be put on a numerical basis. It is easy to show that a good approximation to the contribution of the second peak is

\[
\sqrt{\frac{6}{\pi n}} \left\{ \frac{2}{\sqrt{3}} (\cos x_0)^n \left[ 1 - \left( \frac{1}{4} + \frac{1}{x_0^2} \right) \frac{1}{n} \right] \right\},
\]

where \(x_0\) is the second root of \(\tan x = x\). For \(n = 12\), this gives \(+0.00000 00049 6\), and adding this to the contribution from expression (2) given above, we arrive at the value 0.39392 55651 5, which is in adequate agreement with the true value. The discrepancies at \(n = 13\) and 14, already noted, are accounted for similarly. The correction is negligible for larger \(n\)'s.

We have, accordingly, used formula (2), with the aid of an Hec 2M digital computer, to extend the table in [2] up to a value of \(n\) such that a small number of terms of the formula will suffice to compute further entries. Table 2 covers the range \(30 \leq n \leq 100\). Thereafter, four terms of formula (2) give ten-place accuracy or better.

Returning to the general form \(I_n(b)\), Goddard recommends using, for large \(n\), an expansion derived in the same way as formula (2) above. Goddard's expansion, taken to two more terms than he gives, is

\[
I_n(b) \approx \sqrt{\frac{6}{\pi n}} \exp \left( -\frac{36^2}{2n} \right) \left\{ 1 - \frac{3}{20} \frac{1}{n} \right\} + \left( \frac{9b^2}{10} - \frac{13}{1120} \right) \left( \frac{1}{n} \right)^2
\]

\[
- \left( \frac{9}{20} b^4 + \frac{81}{280} b^2 - \frac{27}{3200} \right) \left( \frac{1}{n} \right)^3
\]

\[
+ \left( \frac{603}{560} b^4 - \frac{243}{1600} b^2 + \frac{52791}{3942400} \right) \left( \frac{1}{n} \right)^4.
\]

The leading term is given without derivation in [6].

Since in the fourth term of the series \(b^4\) appears to a higher power than \(1/n\) (in later terms the largest excess of the power of \(b\) over that of \(1/n\) increases without limit), the approximation is obviously useless for large \(n\) and \(b\) not small, but it may be useful for small \(b\) (say <1), where only limited accuracy is required.

For general \(b\) and large \(n\) we have not found a satisfactory computation procedure. However, for integral \(b\), a suitable recurrence formula is readily derived.
Integration by parts gives

\[ I_n(b) = -\frac{2n}{\pi b} \int_0^\infty \left( \frac{\sin x}{x} \right)^n \cot x \sin (bx) \, dx \]

\[ + \frac{2n}{\pi b} \int_0^\infty \left( \frac{\sin x}{x} \right)^{n+1} \cosec x \sin (bx) \, dx. \]

The following results can be obtained straightforwardly.
\( b \) even:
\[
\cot x \sin (bx) = \cos bx + 2 \cos (b - 2)x + 2 \cos (b - 4)x + \cdots + 2 \cos 2x + 1,
\]
\[
\csc x \sin (bx) = 2 \cos (b - 1)x + 2 \cos (b - 3)x + 2 \cos (b - 5)x + \cdots + 2 \cos x.
\]
\( b \) odd:
\[
\cot x \sin (bx) = \cos bx + 2 \cos (b - 2)x + 2 \cos (b - 4)x + \cdots + 2 \cos x,
\]
\[
\csc x \sin (bx) = 2 \cos (b - 1)x + 2 \cos (b - 3)x + 2 \cos (b - 5)x + \cdots + 2 \cos 2x + 1.
\]

We consequently have the following relations.

\( b \) even:
\[
I_n(b) = -\frac{n}{b} \left[ I_n(b) + 2I_n(b - 2) + 2I_n(b - 4) + \cdots + 2I_n(2) + I_n(0) \right]
\]
\[
+ \frac{n}{b} \left[ 2I_{n+1}(b - 1) + 2I_{n+1}(b - 3) + \cdots + 2I_{n+1}(1) \right].
\]
\( b \) odd:
\[
I_n(b) = -\frac{n}{b} \left[ I_n(b) + 2I_n(b - 2) + 2I_n(b - 4) + \cdots + 2I_n(1) \right]
\]
\[
+ \frac{n}{b} \left[ 2I_{n+1}(b - 1) + 2I_{n+1}(b - 3) + \cdots + 2I_{n+1}(2) + I_{n+1}(0) \right].
\]

Using these relations, from a table of \( I_n(0) \) there can be derived a table of \( I_n(1) \), and then a table of \( I_n(2) \), and so on (at each stage the largest \( n \) is one less than at the previous stage though, in view of the existence of the simple limiting form for \( I_n(0) \) when \( n \) is large, this is no hardship).

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