An Algorithm for Solving a Polynomial Congruence, and its Application to Error-Correcting Codes

By M. H. McAndrew

1. Introduction. The solution of \( f(x) = 0 \) in the \( p \)-adic field may be calculated by the Newton-Raphson process, the iteration of the transformation: \( x \rightarrow x - \frac{f(x)}{f'(x)} \); as in the real field the formula cannot be applied successfully unless we have an initial approximation sufficiently close to a root for the subsequent iteration to converge. (In the \( p \)-adic field, "sufficiently close" is equivalent to "congruent to a sufficiently high power of \( p \).") In this paper we deduce a simple criterion to ensure that the initial approximation is suitable and we develop a procedure for calculating the roots of \( f(x) = 0 \pmod{p^k} \) for any value of \( k \), using the above process where applicable and a single-stepping procedure elsewhere. In §6 we apply this algorithm to investigate solutions of a congruence connected with the existence of close-packed error-correcting binary codes. We deduce that for \( n < 2^{10} \) and \( 2 \leq r \leq 20 \) there are no such codes other than the trivial codes and the Golay code. This result complements results of Shapiro and Slotnick [5] and Selfridge [4] which show that there are no codes for \( r = 2 \), or \( r \) an odd integer less than 135, or \( n < 10^5 \).

2. Notation. \( p \) is a prime and \( f(x) \) a polynomial with integer coefficients; \( f'(x) \) is the formal derivative of \( f(x) \). We use the notation \( p^a \mid B \) for \( p^a \mid B \) and \( p^{a+1} \nmid B \). Define \( l(x) \) by \( p^l \mid f'(x) \). Define

\[
b(m, x) = \max \left\{ \left\lfloor \frac{m + 1}{2} \right\rfloor, m - l(x) \right\}.
\]

We write \( l, l_1, l_2, \ldots \) for \( l(x), l(x_1), l(x_2), \ldots \); similarly, for \( b, b_1, b_2, \ldots \). where the relevant value of \( m \) is clear from the context. We say \( x \) is a solution of type A mod \( p^m \) if

\[
\text{(1) } f(x) = 0 \pmod{p^m}
\]

and \( m \geq 2l + 1 \). We say \( x \) is a solution of type B mod \( p^m \) if (1) holds and \( m \leq 2l \).


Lemma 1. (i) If \( x \) is a solution of type A mod \( p^m \), then \( b = m - l \) and \( 2b \geq m + 1 \geq 2l + 2 \).

(ii) If \( x \) is a solution of type B mod \( p^m \) then \( b = \lfloor (m + 1)/2 \rfloor \) and \( b \leq l \).

Proof. These results follow directly from the definition of solution type.

Lemma 2. If \( f(x) = 0 \pmod{p^m} \) and \( x_1 \equiv x \pmod{p^b} \), then

(i) \( x_1 \) is a solution mod \( p^m \) of the same type as \( x \).

(ii) \( b_1 = b \).

(iii) If \( x \) is of type A mod \( p^m \) then \( l_1 = l \).

Proof. By hypothesis, \( x_1 = x + up^b \) for integral \( u \); hence,

\[
f(x_1) = f(x) + up^bf'(x) + vp^{2b},
\]

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(3) \[ f'(x_1) = f'(x_1) + wp^b, \]
for integral \( v \) and \( w \), by Taylor's theorem for polynomials. Now \( p^m | f(x) \) and, by definition of \( b, b + l \geq m \) and \( 2b \geq m \); hence in (2)

(4) \[ f(x_1) = 0 \pmod{p^m}. \]

To complete the proof we distinguish two cases.

(a) If \( x \) is a solution of type A \( \pmod{p^m} \) then, by Lemma 1 (i), \( b \geq l + 1 \); hence, in (3), \( p^b | f'(x_1) \), i.e., \( l_1 = l \). Therefore \( 2l_1 + 1 = 2l + 1 \leq m, x_1 \) is a solution of type A \( \pmod{p^m} \), and \( b_1 = m - l_1 = m - l = b \).

(b) If \( x \) is a solution of type B \( \pmod{p^m} \) then, by Lemma 1 (ii), \( b \leq l \); hence, in (3), \( l_1 = b = [(m + 1)/2] \), i.e., \( 2l_1 \geq m \). Hence \( x_1 \) is a solution of type B \( \pmod{p^m} \) and \( b_1 = [(m + 1)/2] = b \), by Lemma 1 (ii).

This concludes the proof of Lemma 2.

In view of Lemma 2, we define a solution-set \( \pmod{p^m} \) as the set of all \( x_1 \) with \( x_1 \equiv x \pmod{p^b} \), where \( x \) is a solution of (1) and \( b = b(m, x) \). We use the notation \( (x, b, m) \) for such a solution-set and say \( x \) is a representative of it. By Lemma 2 (ii), the value of \( b \) is independent of the choice of representative and, by Lemma 2 (i), we may define unambiguously the type of a solution-set as the type of any representative. Let \( S(m) \) be the totality of solution-sets \( \pmod{p^m} \).

We define an extension \( \pmod{p^{m+r}} \) of the solution-set \( (x, b, m) \) as a solution-set \( (x_1, b_1, m + r) \) with \( x_1 \equiv x \pmod{p^b} \). Clearly \( S(m + r) \) consists of just all extensions \( \pmod{p^{m+r}} \) of the solution-sets of \( S(m) \).

Theorem 1. (i) If \( (x, b, m) \) is a solution-set of type A, then it has a unique extension, \( (x_1, b_1, m + 1) \) to \( \pmod{p^{m+1}} \); this extension is also of type A with \( l_1 = l \) and \( b_1 = b + 1 \).

(ii) If \( (x, b, m) \) is a solution-set of type B, then (a) if \( m \) is odd either \( (x, b, m + 1) \) is the unique extension of \( (x, b, m) \) to \( \pmod{p^{m+1}} \) or there is no extension to \( \pmod{p^{m+1}} \); (b) if \( m \) is even, the extensions to \( \pmod{p^{m+1}} \) are just those \( (x + sp^b, b + 1, m + 1) \) for which \( 0 \leq s < p \) and \( f(x + sp^b) = 0 \pmod{p^{m+1}} \).

Proof. For any integral \( s \),

(5) \[ f(x + sp^b) = f(x) + sp^bf'(x) + wp^{2s}, \]
for integral \( v \).

(i) If \( x \) is a solution of type A then, by Lemma 1 (i), \( b = m - l \) and \( 2b \geq m + 1 \); hence, from (5), \( f(x + sp^b) \equiv 0 \pmod{p^{m+1}} \) if and only if

(6) \[ p^{-m}f(x) + sp^{-1}f'(x) \equiv 0 \pmod{p}. \]

Since \( p \nmid p^{-m}f'(x) \), (6) has a unique solution \( \pmod{p} \) for \( s, s_0 \) say. Let \( x_1 = x + s_0p^b \); then the unique extension of \( (x_1, b_1, m) \) to \( \pmod{p^{m+1}} \) is clearly \( (x_1, b_1, m + 1) \). Further, \( l_1 = l \), by Lemma 2 (iii); hence \( m + 1 > 2l_1 + 1 \) and so \( (x_1, b_1, m + 1) \) is of type A with \( b_1 = m + 1 - l_1 = m + 1 - l = b + 1 \).

(ii) In this case, by Lemma 1 (ii), \( b = [(m + 1)/2] \). (a) If \( m \) is odd, then \( b = (m + 1)/2 \); hence \( b + l = (m + 1)/2 + l \geq (m + 1)/2 + m/2 > m \). Therefore in (5) \( f(x + sp^b) \equiv f(x) \pmod{p^{m+1}} \). Hence if \( f(x) \not\equiv 0 \pmod{p^{m+1}} \), then \( (x, b, m) \) has no extension to \( \pmod{p^{m+1}} \); if \( f(x) \equiv 0 \pmod{p^{m+1}} \) then, since \( m + 1 \leq 2l, x \) is a solution of type B \( \pmod{p^{m+1}} \) with
\[ b(m + 1, x) = \left\lceil \frac{m + 1 + 1}{2} \right\rceil, \quad \text{by Lemma 1 (ii)} \]
\[ = \frac{m + 1}{2}, \quad \text{since } m \text{ is odd} \]
\[ = b(m, x), \]

i.e., in this case \((x, b, m + 1)\) is the unique extension. (b) If \(m\) is even, then \(b = m/2\).

For any \(s\), \(x + sp^b\) is a solution of type B \(\mod p^m\), by Lemma 2 (i), i.e., \(l' = l(x + sp^b) \geq m/2\). If \(f(x + sp^b) \equiv 0 \mod p^{m+1}\) then

\[ b(m + 1, x + sp^b) = \max \left( \left\lceil \frac{m + 1 + 1}{2} \right\rceil, m + 1 - l' \right) \]
\[ = \max \left( \frac{m + 2}{2}, m + 1 - l' \right) \]
\[ = \frac{m + 2}{2}, \quad \text{since } l' \geq \frac{m}{2}, \]

\[ = b + 1. \]

I.e., the solution-set \(\mod p^{m+1}\) containing \(x + sp^b\) is just \((x + sp^b, b + 1, m + 1)\).

This completes the proof of Theorem 1.

**Theorem 2.** If \((x, b, m)\) is a solution-set of type A then

\( f(x) + uf'(x) \equiv 0 \mod p^{2m-2l} \)

has a solution \(u\), unique \(\mod p^{2m-2l}\), and \((x + u, 2m - 3l, 2m - 2l)\) is the unique extension to \(\mod p^{2m-2l}\) of \((x, b, m)\).

**Proof.** Since \((x, b, m)\) is a solution-set of type A, \(m > 2l\). Hence, since \(p^m | f(x)\) and \(p^l | f'(x)\), equation (7) has a solution for \(u\), unique \(\mod p^{2m-2l}\). Further \(p^{m-l} | u\) since, from (7), \(uf'(x) \equiv 0 \mod p^m\). By Taylor's theorem,

\[ f(x + u) = f(x) + uf'(x) \mod p^{2m-2l} \]
\[ \equiv 0 \mod p^{2m-2l}, \quad \text{by (7)}. \]

Therefore \(x + u\) is a solution \(\mod p^{2m-2l}\) and, since \(p^b = p^{m-l} | u, x + u \in (x, b, m)\).

By Theorem 1 (i) the solution-set \((x, b, m)\) has a unique extension \((x_1, b + 1, m + 1)\) to \(\mod p^{m+1}\), also of type A; by induction it has a unique extension \((x_{m-2l}, b + m - 2l, 2m - 2l)\) to \(\mod 2m - 2l\). Since \(x + u\) is a solution \(\mod p^{2m-2l}\) this concludes the proof of the theorem.

**4. Description of the Algorithm.** The solution-sets of an integral polynomial \(f(x) \mod p^m\) form a tree with extension as the connective. For example, the solution-sets of \(f(x) = (x + 1)(x^2 - x + 6) \mod 2^m\) are depicted in Figure 1. We can construct all the solution-sets by starting with the unique solution-set \(mod p^0\), namely, \((0, 0, 0)\), and calculate the solution-sets \(mod p^{m+1}\) as the extensions of the solution-sets \(mod p^m\). For a solution-set of type A we may construct its extension \(\mod p^N\) in about \(\log_2 N\) steps by the algorithm of Theorem 2. For solution-sets of type B \(\mod p^m\) we construct the solution-sets \(\mod p^{m+1}\) by means of the criteria of Theorem 1.
5. Interpretation in the $p$-adic Field. The solutions of $f(x) \equiv 0$ to arbitrary high powers of $p$ correspond to the solution of $f(x) = 0$ in the $p$-adic field. In this interpretation a solution-set $(x, b, m)$ corresponds to an interval in which $f(x)$ is small in the $p$-adic valuation; specifically, $|f(y)|_p \leq p^{-m}$ for $|y - x|_p \leq p^{-b}$. The relevance of the definition of type of solution-sets is indicated by Theorem 1. If $(x, b, m)$ is a solution-set of type A then, by induction of Theorem 1 (i), there is a unique solution $y$ of $f(y) = 0$ in $|y - x|_p \leq p^{-b}$. On the other hand, if $(x, b, m)$ is a solution-set of type B then although $|f(y)|_p$ is “small” in the range $|y - x|_p \leq p^{-b}$ there may be no solutions of $f(y) = 0$ in this range, or one or more solutions. Theorem 2 exhibits the operation of the Newton-Raphson algorithm. The computation of $-f(x)/f'(x)$ corresponds to solving equation (7) to modulus $p^\infty$. For computational purposes we must be satisfied with solving the equation to modulus some suitably high power of $p$. Restriction of the algorithm to solution-sets of type A both guarantees that the iteration converges (in the $p$-adic topology) and indicates the “right” modulus in which to solve equation (7), namely $p^{b_m-2}$. By “right” we mean that no greater modulus will guarantee a smaller value of $|f(x')|_p$ for the next iterate $x'$.

From the $p$-adic interpretation it also follows that there are no type B solutions for some sufficiently large modulus, unless the rational polynomial $f(x)$ has a repeated factor. For if $(x_n, b, n)$ is a convergent sequence of type B solution-sets then $|f(x_n)|_p \leq p^{-n}$ and $|f'(x_n)|_p \leq p^{-1} \leq p^{-\frac{n}{2}}$. Hence $\lim_{n} x_n$ is a root of both $f(x)$ and $f'(x)$. Further, the existence of a common root of $f(x)$ and $f'(x)$ in the $p$-adic field implies a repeated factor of the rational polynomial $f(x)$ since the two discriminants are formally the same.

6. The Search for Close-Packed Codes. The existence of a close-packed error-correcting binary code [2] requires integers $x, r$ with
The algorithm described in §4 was programmed for the IBM 704 to search for solutions of \( f_r(x) \equiv 0 \pmod{2^m} \). For all \( m, r \) with \( 2 \leq r \leq 20 \) and \( 0 \leq m \leq 139 \) the least value of \( x \) with

\[
0 \leq x < 2^{70},
\]

\( f_r(x) \equiv 0 \pmod{2^m} \)

and

\( f_r(x) \not\equiv 0 \pmod{2^{m+1}} \)

was printed and also an indication of whether or not

\[
x < r \cdot 2^{\lfloor (m+r-1)/r \rfloor}.
\]

Finally it was determined for each value of \( r \) that there were no solutions of \( f_r(x) \equiv 0 \pmod{2^{140}} \) with \( 0 \leq x < 2^{70} \). Now if \( f_r(x) = (r!) \cdot 2^k \) with \( 0 \leq x < 2^{70} \) then either \( k + s \geq 140 \) (where \( 2^s \mid r! \)) or equations (9) hold with \( m = k + s \).

In the latter case inequality (10) must also be satisfied. For if not, then \( x \geq r \cdot 2^{m/r} \) and hence \( f_r(x) \geq (x - r)^r \geq r^r (2^{m/r} - 1)^r \geq r^r (3 \cdot 2^{m/r}/4)^r = (3r/4)^r \cdot 2^m > (r!) \cdot 2^m > (r!) \cdot 2^k \).

The only solutions of (9) and (10) found for \( 2 \leq r \leq 20 \) and \( 2r + 1 < x \) were \( x = 90, r = 2 \) and \( x = 23, r = 3 \). Hence there are no solutions of \( f_r(x) = (r!) \cdot 2^k \) for \( 2 \leq r \leq 20 \) and \( 0 \leq x < 2^{70} \) other than

(i) \( 0 \leq x \leq r \) for arbitrary \( r \); these do not correspond to close-packed codes.

(ii) \( x = 2r + 1 \) for arbitrary \( r \); these correspond to the trivial \( r \) error-correcting codes of two code points of length \( 2r + 1 \).

(iii) \( x = 90, r = 2 \); this does not correspond to a close-packed code as shown in [1].

(iv) \( x = 23, r = 3 \); this corresponds to the Golay-Paige code of \( 2^{12} \) code points of length 23 [1, 3].

IBM Watson Research Center
Yorktown Heights, New York