Mantissa Distributions

By Alan G. Konheim

Let \( b \) be an integer, at least 2, and write each positive real number in the form

\[ x = mb^c, \]

where \( m \) (the mantissa) satisfies \( 1/b \leq m < 1 \) and \( c \) (the characteristic) is an integer.

We define the product of mantissas* \( m_1 \) and \( m_2 \) by

\[ m_1 \cdot m_2 = \begin{cases} m_1 m_2 & \text{if } 1/b \leq m_1 m_2 < 1, \\ bm_1 m_2 & \text{if } 1/b^2 \leq m_1 m_2 < 1/b. \end{cases} \]

Now suppose that \( M_1 \) and \( M_2 \) are independent, identically distributed random variables, each taking on values in the interval \([1/b, 1)\) such that

\[ \Pr(M_1 \cdot M_2 \leq x) = \Pr(M_1 \leq x). \]

What are all of the possible choices for the distribution function of \( M_1 \)? The answer is provided by the following

**Theorem.** \( \Pr(M_1 \leq x) = F_n(x) \) or \( F_\infty(x) \) \( (n = 1, 2, \ldots) \), where

\[ F_n(x) = \begin{cases} 0 & \text{if } -\infty < x < b^{-1}, \\ 1/n & \text{if } b^{-1} \leq x < b^{-1+1/n}, \\ 2/n & \text{if } b^{-1+1/n} \leq x < b^{-1+2/n}, \\ \vdots & \\ 1 & \text{if } b^{-1} \leq x < \infty, \\ 0 & \text{if } -\infty < x < b^{-1}, \end{cases} \]

\[ F_\infty(x) = \begin{cases} 0 & \text{if } -\infty < x < b^{-1}, \\ 1 + 1/n \left[ \frac{\log x}{\log b} + 1 \right] & \text{if } b^{-1} \leq x < 1, \\ 1 & \text{if } 1 \leq x < \infty, \end{cases} \]

and

\[ F_\infty(x) = 1 + \frac{\log x}{\log b} = \int_{1/b}^{x} \frac{du}{u \log b} \text{ if } b^{-1} \leq x < 1, \]

\[ \int_{1/b}^{x} \frac{du}{u \log b} \]

**Proof.** We will write \( M_1 = b^{-\Theta_1} \) \((i = 1, 2)\), where \( \Theta_1 \) and \( \Theta_2 \) are independent, identically distributed random variables, taking on values in \((0, 1]\). Note that

\[ M_1 \cdot M_2 = b^{-(\Theta_1 + \Theta_2)}, \]

Received June 22, 1964.

* If \( m_i \) is the mantissa of \( x_i \) then \( m_1 \cdot m_2 \) is the mantissa of \( x_1x_2 \).

† \([ ]\) denotes 'the integer part of.'

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
where \( \oplus \) denotes addition modulo one. Thus (3) is equivalent to requiring that \( \Theta_1 \oplus \Theta_2 \) and \( \Theta_1 \) have the same distribution. If we set

\[
\phi(n) = E\{e^{2\pi i n \Theta_1}\} = \int_0^1 e^{2\pi i n \theta} \ dF_{\Theta_1}(\theta),
\]

then (3) and the independence of \( \Theta_1, \Theta_2 \) imply

\[
\phi(n) = E\{e^{2\pi i (\Theta_1 + \Theta_2)}\} = E\{e^{2\pi i (\Theta_1 + \Theta_2)}\} = \phi^2(n)
\]

so that \( \phi(n) = 0 \) or 1. Certainly \( \phi(0) = 1 \). There are two cases to be examined.

Case 1. \( \phi(n) = 0 \) for all \( n \neq 0 \).

It follows from the uniqueness theorem for Fourier-Stieltjes series that

\[
dF_{\Theta_1}(d\theta_1) = d\theta_1 \quad \text{and hence} \quad \Pr(M_1 \leq x) = F_{\alpha}(x).
\]

Case 2. \( \phi(n) = 1 \) for some \( n \neq 0 \).

Let \( m \) be the smallest positive integer such that \( \phi(m) = 1 \). Then

\[
0 = \int_0^1 (1 - e^{2\pi i \theta m}) \ dF_{\Theta_1}(\theta) = \int_0^1 (1 - \cos 2\pi m \theta) \ dF_{\Theta_1}(\theta).
\]

It follows that \( F_{\Theta_1} \) is a 'step function' with points of discontinuity at \( \theta_k = k/m \) \( (k = 1, 2, \ldots, m) \) and, hence, \( \phi(n + m) = \phi(n) \) \( (n = 0, \pm 1, \pm 2, \ldots) \). We assert that \( \phi(n) = 1 \) if and only if \( n = km \) for some integer \( k \); for if \( \phi(n) = 1 \) with \( km < n < (k + 1)m \) then \( \phi(n - km) = \phi(n) = 1 \) while \( 0 < n - km < m \) contradicting the minimality of \( m \). The uniqueness theorem for Fourier-Stieltjes series now shows that \( \Pr(M_1 \leq x) = F_{m}(x) \).

I should like to acknowledge with thanks several suggestions made by Mr. Benjamin Weiss.

Thomas J. Watson Research Center
International Business Machines Corporation
Yorktown Heights, New York

New Primes of the Form \( n^4 + 1 \)

By A. Gloden

This note presents some results of a continuation of the author's systematic factorization of integers of the form \( n^4 + 1 \) [1].

An electronic computer at l'Institut Blaise Pascal in Paris has been used to find solutions of the congruence \( x^4 + 1 \equiv 0 \pmod{p} \) for all primes of the form \( 8k + 1 \) in the interval \( 10^6 < p < 4 \cdot 10^6 \), thereby extending the previous range of such tables listed in [1].

With the aid of these tables, the complete factorization of \( n^4 + 1 \) has now been effected for all even values of \( n \) less than 2040 and for all odd values less than 2397.

Thus, the primality of \( \frac{1}{2}(n^4 + 1) \) has been established for the following 116 values of \( n \):