 Estimates of Weights in Gauss-Type Quadrature

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1. Introduction. It may readily be verified that the angular distance
\( \Delta \theta = \theta_{i+1,n} - \theta_{i,n} \) between the zeros \( \theta_{i,n} \) of the Legendre polynomial \( P_n(\cos \theta) \) in \( \cos \theta \) is roughly constant for large \( n \). From the quadrature formula itself the weights may be estimated to a corresponding degree of accuracy. Direct asymptotic estimates of the weights corresponding to \( \cos \theta = 0 \) in the \((2n + 1)\)-point Gaussian quadrature are all available from Stirling's formula in the cases considered below. We here replace the \( P_n \) by \( C_n^{\lambda} \), the Gegenbauer polynomials (effectively, tesseral harmonics or ultraspherical polynomials) of order \( \lambda \geq 0 \), and the \( H_n \) in the single limiting set of Hermite polynomials. Explicit formulas are derived: but the estimates for the general weights have a precision limited by the corresponding precision of the estimates of the zeros.

2. The Quadrature Formula. The Lagrange interpolation formula
\[
f(x) = \sum_i \frac{P(x)f(x_i)}{P'(x_i)(x - x_i)}, \quad P(x_i) = 0,
\]
(1)
\( P'(x_i) \neq 0 \), \( i = 1, 2, \ldots, n \),
algebraically valid for polynomials \( f \) of degree \( \nu < n \), the degree of \( P \), has a rather limited direct use in polynomial approximation theory. Combined with various restrictions on \( P \) to be in a basis of a set of polynomials with suitable properties, it becomes more useful.

Let \( P^*(x) \) be of degree \( n + 1 \), so that \( P^*(x) = axP(x) - bP(x) - cP^*(x) \) for constants \( a, b, \) and \( c \), \( P^* \) representing a polynomial of degree \( \nu < n \). We set
\[
K(x, t) = K(t, x) = \frac{P^*(x)P(t) - P^*(t)P(x)}{x - t},
\]
a polynomial of degree \( n \) in \( x \) for each \( t \), so that
\( K(x, t) = aP(x)P(t) + cK^*(x, t) \),
\( K^* \) being defined in terms of \( P \) and \( P^* \) exactly as \( K \) is determined by \( P^* \) and \( P \). In particular, \( K(x, x) = P(x)P^*(x) - P^*(x)P'(x) \); and (1) is modified to become
\[
f(x) = \sum_i \frac{K(x, x_i)}{K(x_i, x_i)} f(x_i).
\]
(2)
A suitable normalization with respect to a fixed integrable weight function \( w \), essentially positive over the interval \( I \) of integration, is
\[
\int_I K(x, x_i)w(x) \, dx = 1,
\]
so that (2) becomes
\[
\int_I f(x)w(x) \, dx = \sum_i f(x_i)W_i,
\]
(3)
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where

\[(4) \quad W_i = \frac{1}{K(x_i, x_i)} \]

is the formula for the weights.

From the above,

\[K_n(x, t) = \sum_{i=0}^{n} a_i p_i(x) p_j(t),\]

the indices \(j\) indicating the degrees of the polynomials \(p_j\). Referring to (1), for example, we set

\[p_n(t) = k_n t^n - \sum_{i=0}^{n-1} c_{j,n} p_j(t), \quad n = 1, 2, 3, \ldots,\]

where

\[p_0(t) = k_0 > 0, \quad \int_I w(t) \, dt = \frac{1}{k_0^2}, \]

\[(5) \quad \int_I p_n(t) p_j(t) w(t) \, dt = 0, \quad 0 \leq j < n,\]

and

\[\int_I |p_n(t)|^2 w(t) \, dt = 1.\]

The inductive definition is complete if we assume \(k_n > 0\). Indeed, for an arbitrary polynomial \(P\),

\[(5') \quad P(t) = \sum_{j=0}^{n} a_j k_j t^j = \sum_{j=0}^{n} a_{j,n} p_j(t),\]

the \(a_{j,n}\) being determined uniquely by the \(a_j\) and \(k_j\), where \(\int_I k_j t^j p_j(t) w(t) \, dt = 1\), so that

\[(6) \quad xp_n(x) = \frac{k_n}{k_{n+1}} p_{n+1}(x) + b_n p_n(x) + \frac{k_{n-1}}{k_n} p_{n-1}(x) + \sum_{j=0}^{n-2} b_{j,n} p_j(x)\]

in any case, with \(b_{j,n} = 0\) by (5). Then

\[K_n(x, t) = \sum_{j=0}^{n} p_j(x) p_j(t) = \frac{k_n}{k_{n+1}} \frac{p_{n+1}(x) p_n(t) - p_n(x) p_{n+1}(t)}{x - t}, \quad \text{and}\]

\[(7) \quad K_n(x, x) = \sum_{j=0}^{n} \{p_j(x)\}^2 = \frac{k_n}{k_{n+1}} \{p_{n+1}(x) p_n(x) - p_n'(x) p_{n+1}(x)\} = \int_I \{K_n(x, t)\}^2 w(t) \, dt,\]

these being the standard Christoffel formulae (see [1]).
If \( f \) is of degree \( 2n - 1 \) or less, the quotient \( Q \) of \( f \) by \( p_n \) is uniquely determined, with remainder \( p_\star(t) = f(t) - Q(t)p_n(t) \) of degree \( n - 1 \) or less. Then, if \( p_n(x_i) = 0 \), \( n \) being fixed,

\[
\int f(t)w(t) \, dt = \int f(t)w(t) \, dt, \quad \text{by (5), and}
\]

\[
\int f(t)w(t) \, dt = \sum_i W_i f(x_i),
\]

as before. (The formulae (7) guarantee the separation of \( n \) distinct zeros in \( I \).)

3. Sums of Squares. The Cesàro-one sums

\[
\sigma_n(x, t) = \frac{1}{n} \sum_{j=0}^{n-1} K_j(x, t),
\]

are expressed in the way suggested by Christoffel's method as follows:

\[
n(x - t)^2 \sigma_n(x, t) = \sum_{j=0}^{n-1} k_j \; (b_j - b_{j+1}) \{ p_{j+1}(x)p_j(t) + p_{j+1}(t)p_j(x) \}
\]

\[
+ \frac{k_{n-1}}{k_{n+1}} \{ p_{n+1}(x)p_{n-1}(t) + p_{n-1}(x)p_{n+1}(t) \}
\]

\[
- 2 \left( \frac{k_{n-1}}{k_n} \right)^2 p_n(x)p_n(t) + 2 \sum_{j=0}^{n-1} p_j(x)p_j(t) \left\{ \left( \frac{k_j}{k_{j+1}} \right)^2 - \left( \frac{k_{j-1}}{k_j} \right)^2 \right\},
\]

where \( b_j = \int t[p_j(t)]^2 w(t) \, dt \) and \( k_{-1} = 0 \).

Beginning with \( k_2(b_1 - b_0)/k_1 = c_{1,2} \), we see that \( b_j = b_{j+1} \) for all \( j \) if and only if \( w \) is symmetric over \( I \). After a translation, we may assume in this case that the \( p_i(t) \) are alternately even and odd polynomials. We assume that this condition holds in the sequel.

Let

\[
\Lambda_j(x) = \frac{k_{j-1}}{k_j} p_{j-1}(x) - \frac{k_j}{k_{j+1}} p_{j+1}(x),
\]

so that

\[
4 \frac{k_{j-1}}{k_{j+1}} p_{j+1}(x)p_{j-1}(x) = x^2\{ p_j(x) \}^2 - \{ \Lambda_j(x) \}^2.
\]

Then, for suitable constants \( c_n \), we set

\[
L_n(x) = \left( c_n^2 - x^2 \right) \{ p_n(x) \}^2 + \{ \Lambda_n(x) \}^2
\]

\[
= 4 \sum_{j=0}^{n-1} \{ p_j(x) \}^2 \left\{ \left( \frac{k_j}{k_{j+1}} \right)^2 - \left( \frac{k_{j-1}}{k_j} \right)^2 \right\} + c_n^2 - 4 \left( \frac{k_{n-1}}{k_n} \right)^2 \{ p_n(x) \}^2.
\]

To make this formulation of sums of squares useful, the weight function \( w \) is further restricted.

4. Gegenbauer Polynomials. See [1].

The expansion of \( \rho^{-2\lambda} = (1 - 2rt + r^2)^{-\lambda} \) as a power series in \( r \),

\[
(1 - rz)^{-\lambda}(1 - r^2)^{-\lambda} = \sum_{j=0}^{\infty} C_j^\lambda(t)r^j,
\]

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subject to

\[ z + \bar{z} = 2t = 2 \cos \theta, \quad \bar{z} = 1, \quad 0 \leq r < 1, \]

determines the Gegenbauer polynomials \( C_n^\lambda \) of order \( \lambda > 0 \). If \( y \) is any successively differentiable function of \( \rho \),

\[ r^2 \frac{\partial^2 y}{\partial r^2} + \frac{\partial^2 y}{\partial t^2} = r^2 \frac{\partial^2 y}{\partial \rho^2}. \]

In the above case, \( y = \rho^{-2\lambda} \), so \( \frac{d^2 y}{d\rho^2} + \left( (2\lambda + 1)/\rho \right) \left( \frac{dy}{d\rho} \right) = 0 \), and so

\[ r^2 \frac{\partial^2 y}{\partial r^2} + (2\lambda + 1)r \frac{\partial y}{\partial r} + (1 - t^2) \frac{\partial^2 y}{\partial t^2} = (2\lambda + 1)t \frac{\partial y}{\partial t}. \]

Comparing coefficients in the power series, we have

\[ \frac{d}{dt} \left\{ (1 - t^2)^{-\lambda+1/2} \frac{dC_n^\lambda(t)}{dt} \right\} = -n(n + 2\lambda)(1 - t^2)^{-\lambda-1/2}C_n^\lambda(t). \]

Multiplying by \( C_j^\lambda(t) \), alternating the indices \( n \) and \( j \), and subtracting, then integrating from \( t = -1 \) to \( t = 1 \), we have

\[ C_j^\lambda(t) = \sqrt{h_j} p_j(t), \]

the \( \{p_j\} \) being orthogonal (with property (5)) with respect to \( w \),

\[ w(t) = (1 - t^2)^{\lambda-1/2}. \]

Here,

\[ \int_{-1}^{+1} \left\{ C_j^\lambda(t) \right\}^2 w(t) \, dt = h_j, \]

easily calculated explicitly. From the definition above, using the series and the binomial theorem,

\[ C_n^\lambda(\cos \theta) = \sum_{j=0}^{n} \binom{\lambda + j - 1}{j} \binom{\lambda + n - j - 1}{n - j} \cos(n - 2j \theta), \]

so

\[ |C_n^\lambda(t)| \leq C_n^\lambda(1) = \binom{2\lambda + n - 1}{n}, \quad -1 \leq t \leq 1, \]

if \( \lambda > 0 \).

We may make direct use of the Christoffel formulae (7), comparison of terms in a linear expansion, and induction, to obtain

\[ 2h_n k_0^2 (n + \lambda) = \lambda \binom{n + 2\lambda - 1}{n}, \]

\[ 4 \left( \frac{k_{n-1}}{k_n} \right)^2 = \frac{n(n + 2\lambda - 1)}{(n + \lambda)(n - 1 + \lambda)}, \]
and
\[ (12) \quad \lambda^2 2^{2\lambda}(n + \lambda)|C_n^\lambda(t)|^2 = \pi \binom{2\lambda + n - 1}{n} \lambda(2\lambda)! |p_n(t)|^2. \]

Also,
\[ \frac{k_{n-1}}{k_n} p_{n-1}(0) = -\frac{k_n}{k_{n+1}} p_{n+1}(0), \]
so that
\[ (13) \quad \lim_{n \to \infty} |p_{2n}(0)|^2 = \frac{2}{\pi}, \]
and
\[ \lim_{n \to \infty} p_n(1)(n + \lambda)^{-2\lambda} = \sqrt{\frac{2}{\pi}} \frac{2^{2\lambda}!}{(2\lambda)!}, \]
the relative errors in the corresponding approximations being of (order) \( O(1/(n + \lambda)^2) \) uniformly in \( n \) for fixed \( \lambda \) by Stirling's formula.

We set \( \tau = (1 - t^2)^{1/2} p_n(t) \), and find
\[ \frac{d\tau}{dt} = (n + \lambda)(1 - t^2)^{1/2} \Lambda_n(t), \]
using (6) and (11). If
\[ L_n(t) = \{p_n(t)|^2(1 - t^2) + \{\Lambda_n(t)\}^2, \]
(11) becomes
\[ (14) \quad \frac{d}{dt} |L_n(t)(1 - t^2)^{\lambda-1}| = \frac{2\lambda(1 - \lambda)}{n + \lambda} (1 - t^2)^{\lambda-2} p_n(t)\Lambda_n(t). \]

From the above quadratic relation, and (6),
\[ 2 \sqrt{(1 - t^2)} \leq p_n(t)\Lambda_n(t) \leq L_n(t). \]
Differentiating the logarithm of \( L_n \), and integrating, we have
\[ \log \left( \frac{L_n(t)}{L_n(0)} \right) (1 - t^2)^{-\lambda-1} < \frac{\lambda(1 - \lambda)}{n + \lambda} \frac{|t|}{\sqrt{1 - t^2}}, \quad 0 < |t| < 1. \]
In particular, \( \lim_{n \to \infty} L_n(t)(1 - t^2)^{\lambda-1} = 2/\pi, -1 < t < 1. \)
However, relation (10) now reads as follows:
\[ L_n(t) = -2 \sum_{j=0}^{n-1} \frac{\lambda(1 - \lambda)|p_j(t)|^2}{(j + 1 + \lambda)(j + 1 + \lambda)} - \frac{\lambda(1 - \lambda)|p_n(t)|^2}{(n + \lambda - 1)(n + \lambda)}, \]
whence
\[ L_n(t)(1 - t^2)^{-\lambda-1} = \{p_n(t)|^2(1 - t^2)^{\lambda} + \{\Lambda_n(t)\}^2(1 - t^2)^{\lambda-1} \]
\[ (15) \quad = \frac{2}{\pi} \frac{\lambda(1 - \lambda)|p_n(t)|^2(1 - t^2)^{\lambda-1}}{(n + \lambda)(n + \lambda + 1)} + 2 \sum_{j=n+1}^{\infty} \frac{\lambda(1 - \lambda)|p_j(t)|^2(1 - t^2)^{\lambda-1}}{(j + 1 + \lambda)(j + 1 + \lambda + 1)}. \]
The maximum of $z^2 = \{p_n(t)\}^2(1 - t^2)^{2\lambda}$ in any subinterval of $I$ with endpoints $t = x_i$ or $t = \pm 1$, corresponds only to $\Lambda_n(t) = 0$, so that if

$$(n + \lambda)(n + \lambda + 1)(1 - t^2) \geq \frac{|\lambda(1 - \lambda)|}{\varepsilon},$$

$$p_n(t)(1 - t^2)^{\lambda}(1 \pm \varepsilon) < \frac{2}{\pi},$$

and, otherwise,

$$p_n(1 - t^2)^{\lambda}$$

is uniformly bounded, by (12) and (13).

On the other hand, if $p_n(x_i) = 0$,

$$(\lambda(x_i))^2(1 - x_i^2)^{\lambda - 1} = \frac{2}{\pi} + \frac{2\lambda(1 - \lambda)}{1 - x_i^2} \sum_{j=n+1}^{\infty} \frac{|p_j(x_i)|^2(1 - x_i^2)^{\lambda}}{(j - 1 + \lambda)(j + \lambda)(j + 1 + \lambda)},$$

where

$$\sum_{j=n+1}^{\infty} \frac{|p_j(x)|^2(1 - x^2)^{\lambda}}{(j - 1 + \lambda)(j + \lambda)(j + 1 + \lambda)} < \frac{1 + \epsilon_n'}{\pi(n + \lambda)^2}$$

and $\lim_{n \to \infty} \epsilon_n' = 0$, if $|\pm 1 + x| > \delta$, any fixed positive number. That is, if $|\pm 1 + x_i| > \delta$,

$$\frac{1}{W_i} = K_n(x_i, x_i) = \frac{\Lambda_n(x_i)p_n'(x_i)}{2} = \frac{n + \lambda}{2} \frac{\{\Lambda_n(x_i)\}^2}{1 - x_i^2},$$

and for such zeros $x = x_i$,

$$W_i \approx \frac{\pi}{n + \lambda} (1 - x_i^2)^{\lambda},$$

with a relative-error estimate

$$\frac{|\lambda(1 - \lambda)|}{(n + \lambda)^2(1 - x_i^2)}$$

for both upper and lower bounds.

If $n$ is an odd number, and $x_i = 0$, we easily compute

$$\frac{1}{W_i} = \frac{n + \lambda}{\pi} \left\{1 + \frac{\lambda(1 - \lambda)}{2n^2} + \frac{\lambda(1 - \lambda)^2}{n^3} + \cdots\right\},$$

using Stirling's formula, for the corresponding median weight $W_i$. The precision of the estimate here is easily controlled; but in the general case the sums of squares seem difficult to handle with precision.

5. Spacing of Zeros. Let $v = p_n'(t)/p_n(t)$. Using (11), we find

$$(1 - t^2) \frac{dv}{dt} = (2\lambda + 1)tv - n(n + 2\lambda) - (1 - t^2)v^2.$$
$x = x_n$ being the zero of $p_n(t)$ nearest $t = 1$. Since $(p_n(x_n) - p_n(1))/(x_n - 1) < p_n'(1)$, we have $x_n < 1 - (2\lambda + 1)/(n(n + 2\lambda))$.

In general, if we set $t = \sin \phi$, the equivalent differential relation

$$
-\frac{d}{d\phi} \left\{ \arctan \left[ \frac{\Delta_n(t)}{p_n(t)\sqrt{1 - t^2}} \right] \right\} = \frac{n + \lambda}{n + \lambda} \frac{\{p_n(t)\}^2}{L_n(t)}, \quad x_i < t < x_{i+1},
$$
gives us the necessary information concerning the spacing of the zeros. We have

$$
\pi = \Delta \arctan \left[ \frac{\Delta_n(t)}{p_n(t)\sqrt{1 - t^2}} \right] = (n + \lambda)\Delta \phi_i + \frac{\lambda(1 - \lambda)}{n + \lambda} \int_{\phi_i}^{\phi_{i+1}} \frac{\{p_n(t)\}^2}{L_n(t)} d\phi,
$$

where $x_i = \sin \phi_i$ and $\Delta \phi_i = \phi_{i+1} - \phi_i$.

6. Hermite Polynomials. From the defining formulas, we easily obtain

$$
\left( \frac{d}{dt} \right)^m \{C_n^\lambda(t)\} = 2^n \binom{\lambda + m - 1}{m} C_n^{\lambda + m}(t)
$$

by induction on $m$. Among other results, relations between the tesseral harmonics of Legendre,

$$
P_n^{(m)}(t) = (1 - t^2)^{m/2} \left( \frac{d}{dt} \right)^m \{P_n(t)\},
$$

$$
P_n(t) = C_n^\lambda(t) \quad \text{for} \quad \lambda = \frac{1}{2},
$$

and the Gegenbauer polynomials follow. Formally, the trigonometric basis is given by $\lambda = 0$ and $\lambda = 1$.

If $t^2 = s^2/2\lambda$, $s$ being fixed, and $\lambda \to \infty$, we have

$$
w(t) \to e^{-t^2/2}.
$$

For the bounded $n$ and $s$,

$$
C_n^\lambda(t) \to H_n(s), \quad \text{if} \quad \lambda \to \infty,
$$

the corresponding Hermite polynomial.

Let

$$
\frac{d}{dt} \{H_n(t)e^{-t^2/2}\} = -H_{n+1}(t)e^{-t^2/2}, \quad H_0(t) = 1,
$$

for $n = 0, 1, 2, \cdots$. Then

$$
H_n'(t) = nH_{n-1}(t),
$$

by Leibnitz' rule for successive differentiation. It follows immediately that

$$
H_n(x) = \sum_{j<(n+1)/2} \binom{n}{2j} (-1)^j C_j x^{n-2j}
$$
for a single set of coefficients \( \{ C_i \} \). Since
\[
t H_n(t) = n H_{n-1}(t) + H_{n+1}(t)
\]
from the pair of relations given above, we have the Christoffel formulae
\[
H_n(x, t) = \sum_{j=0}^{n} \frac{H_j(x) H_j(t)}{j!} = \frac{H_{n+1}(x) H_n(t) - H_n(x) H_{n+1}(t)}{n!(x - t)}
\]
and
\[
H_n(x, x) = \sum_{j=0}^{n} \frac{H_j^2(x)}{j!} = \frac{(n + 1)H_n^2(x) - n H_{n+1}(x) H_{n-1}(x)}{n!}
\]
\[
= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} H_n^2(x, t) e^{-t^2/2} dt.
\]
To arrive at the last result, we make use of
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2 + t^2)/2} dxdt = 2\pi,
\]
or the limits given above. Since
\[
\sqrt{(2\pi)} e^{-x^2/2} = \int_{-\infty}^{\infty} e^{-t^2/2 + ixt} dt
\]
we have
\[
\sqrt{(2\pi)} H_n(x) e^{-x^2/2} = (-i)^n \int_{-\infty}^{\infty} e^{-t^2/2 + ixt} t^n dt.
\]
Let \( z = e^{-t^2/4} H_n(t) \), so that
\[
\frac{dz}{dt} = e^{-t^2/4} \left\{ n H_{n-1}(t) - \frac{t}{2} H_n(t) \right\}
\]
and
\[
\frac{d^2 z}{dt^2} = -z \left( n + 1 - \frac{t^2}{4} \right).
\]
Then
\[
(tz)^2 - 4 \left( \frac{dz}{dt} \right)^2 = 4ne^{-t^2/2} H_{n-1}(t) H_{n+1}(t),
\]
so that
\[
e^{-x^2/2} \left\{ \sum_{j=0}^{n-1} \frac{H_j^2(x)}{j!} + \frac{1}{2} \frac{H_{n}^2(x)}{n!} \right\} = \sum_{j=0}^{n-1} \frac{H_j^2(0)}{j!} + \frac{1}{2} \frac{H_n^2(0)}{n!} - \frac{1}{2} \int_{0}^{x} te^{-t^2/2} \frac{H_n^2(t)}{n!} dt
\]
from the Christoffel formula. We do not obtain different results from the formu-lation of the Cesàro-one sums, in this case. We define
\[
L_n(x) = \frac{1}{\sqrt{n}} \left\{ \sum_{j=0}^{n-1} \frac{H_j^2(x)}{j!} + \frac{1}{2} \frac{H_n^2(x)}{n!} \right\},
\]
so that

$$\lim_{n \to \infty} L_n(0) = \sqrt{\frac{2}{\pi}}.$$ 

Then, also,

$$L_n(x)e^{-x^2/2} = L_n(0) - \frac{1}{2\sqrt{n}} \int_0^x t \frac{H_n^2(t)}{n!} e^{-t^2/2} dt,$$

so here

$$\sqrt{n}L_n(t)e^{-t^2/2} = \frac{1}{n^2} \left\{ \left( \frac{dz}{dt} \right)^2 + \left( n + \frac{1}{2} - \frac{t^2}{4} \right) z^2 \right\},$$

and

$$\lim_{n \to \infty} L_n(x)e^{-x^2/2} = \sqrt{\frac{2}{\pi}}.$$ 

The formula for the weights $W_i$ corresponding to $H_n(x_i) = 0$ becomes

$$\frac{1}{W_i} = H_n(x_i, x_i) = \sqrt{n} L_n(x_i),$$

so

$$W_i \approx \sqrt{\frac{\pi}{2n}} e^{-x_i^2/2},$$

with a relative error estimate

$$\frac{x_i^2}{2n - \delta} \quad \text{if} \quad x_i^2 < 2(1 + \delta).$$

If we consider the Fourier sine expansion over the interval $(a, a + \pi/k)$ between zeros $x = a, x = b = a + \pi/k$, of $H_n(x)e^{-x^2/4}$, we have

$$\int_a^b \left\{ \left( \frac{dz}{dt} \right)^2 - k^2 z^2 \right\} dt > 0.$$ 

Now

$$\int_a^b \left\{ \left( \frac{dz}{dt} \right)^2 - \left( n + \frac{1}{2} - \frac{t^2}{4} \right) z^2 \right\} dt = 0,$$

so that

$$b - a > \frac{2\pi}{\sqrt{(4n + 2 - a^2)}}.$$ 

Otherwise, $dz/dt < 0$ if $t^2 \geq 4n + 2$. We cannot have $z = 0$ there, since $z > 0$ if $t \to \infty$ for fixed $n$. Then

$$b^2 < 4n + 2.$$
We may point out that the estimates, for Cesàro-one and related sums, remain useful in establishing convergence properties of the expansions of functions (e.g., of bounded variation) as series of orthogonal polynomials.

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