Estimates of Weights in Gauss-Type Quadrature

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1. Introduction. It may readily be verified that the angular distance \( \Delta \theta = \theta_{i+1,n} - \theta_{i,n} \) between the zeros \( \theta_{i,n} \) of the Legendre polynomial \( P_n(\cos \theta) \) in \( \cos \theta \) is roughly constant for large \( n \). From the quadrature formula itself the weights may be estimated to a corresponding degree of accuracy. Direct asymptotic estimates of the weights corresponding to \( \cos \theta = 0 \) in the \((2n + 1)\)-point Gaussian quadrature are all available from Stirling's formula in the cases considered below. We here replace the \( P_n \) by \( C_n^\lambda \), the Gegenbauer polynomials (effectively, tesseral harmonics or ultraspherical polynomials) of order \( \lambda \geq 0 \), and the \( H_n \) in the single limiting set of Hermite polynomials. Explicit formulas are derived: but the estimates for the general weights have a precision limited by the corresponding precision of the estimates of the zeros.

2. The Quadrature Formula. The Lagrange interpolation formula

\[
f(x) = \sum_i \frac{P(x)f(x_i)}{P'(x_i)(x - x_i)}, \quad P(x_i) = 0,\]

\[
P'(x_i) \neq 0, \quad i = 1, 2, \cdots, n,
\]

algebraically valid for polynomials \( f \) of degree \( v < n \), the degree of \( P \), has a rather limited direct use in polynomial approximation theory. Combined with various restrictions on \( P \) to be in a basis of a set of polynomials with suitable properties, it becomes more useful.

Let \( P^*(x) = axP(x) - bP(x) - cP^*(x) \) for constants \( a, b, \) and \( c \), \( P^* \) representing a polynomial of degree \( v < n \). We set

\[
K(x, t) = K(t, x) = \frac{P^*(x)P(t) - P^*(t)P(x)}{x - t},
\]

a polynomial of degree \( n \) in \( x \) for each \( t \), so that

\[
K(x, t) = aP(x)P(t) + cK^*(x, t),
\]

\( K^* \) being defined in terms of \( P \) and \( P^* \) exactly as \( K \) is determined by \( P^* \) and \( P \). In particular, \( K(x, x) = P(x)P^*'(x) - P^*(x)P'(x) \); and (1) is modified to become

\[
f(x) = \sum_i \frac{K(x, x_i)}{K(x_i, x_i)} f(x_i).
\]

A suitable normalization with respect to a fixed integrable weight function \( w \), essentially positive over the interval \( I \) of integration, is

\[
\int_I K(x, x_i)w(x) \, dx = 1,
\]

so that (2) becomes

\[
\int_I f(x)w(x) \, dx = \sum_i f(x_i)W_i,
\]

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where

\[(4) \quad W_i = \frac{1}{K(x_i, x_i)}\]

is the formula for the weights.

From the above,

\[K_n(x, t) = \sum_{i=0}^{n} a_i p_i(x) p_j(t),\]

the indices \(j\) indicating the degrees of the polynomials \(p_j\). Referring to (1), for example, we set

\[p_n(t) = k_n t^n - \sum_{i=0}^{n-1} c_{j,n} p_i(t), \quad n = 1, 2, 3, \ldots,\]

where

\[p_0(t) = k_0 > 0, \quad \int_t w(t) \, dt = \frac{1}{k_0^2},\]

\[(5) \quad \int_t p_n(t) p_j(t) w(t) \, dt = 0, \quad 0 \leq j < n,\]

and

\[\int_t [p_n(t)]^2 w(t) \, dt = 1.\]

The inductive definition is complete if we assume \(k_n > 0\). Indeed, for an arbitrary polynomial \(P\),

\[(5') \quad P(t) = \sum_{j=0}^{n} a_j k_0^{j} = \sum_{j=0}^{n} a_j, \quad p_j(t),\]

the \(a_{j,n}\) being determined uniquely by the \(a_j\) and \(k_j\), where \(\int_t k_j t^j p_j(t) w(t) \, dt = 1\),

so that

\[(6) \quad x p_n(x) = \frac{k_n}{k_{n+1}} p_{n+1}(x) + b_n p_n(x) + \frac{k_{n-1}}{k_n} p_{n-1}(x) + \sum_{j=0}^{n-2} b_{j,n} p_j(x)\]

in any case, with \(b_{j,n} = 0\) by (5). Then

\[K_n(x, t) = \sum_{j=0}^{n} p_j(x) p_j(t)\]

\[= \frac{k_n}{k_{n+1}} \frac{p_{n+1}(x) p_n(t) - p_n(x) p_{n+1}(t)}{x - t}, \quad \text{and}\]

\[(7) \quad K_n(x, x) = \sum_{j=0}^{n} [p_j(x)]^2\]

\[= \frac{k_n}{k_{n+1}} [p_{n+1}(x) p_n(x) - p_n(x) p_{n+1}(x)]\]

\[= \int_t [K_n(x, t)]^2 w(t) \, dt,\]

these being the standard Christoffel formulae (see [1]).
If $f$ is of degree $2n - 1$ or less, the quotient $Q$ of $f$ by $p_n$ is uniquely determined, with remainder $p_*(t) = f(t) - Q(t)p_n(t)$ of degree $n - 1$ or less. Then, if $p_n(x_i) = 0$, $n$ being fixed,
\[ \int_I p_*(t)w(t) \, dt = \int_I f(t)w(t) \, dt, \quad \text{by (5)}, \quad \text{and} \]
\[ \int_I f(t)w(t) \, dt = \sum_i W_if(x_i), \]
as before. (The formulae (7) guarantee the separation of $n$ distinct zeros in $I$.)

3. Sums of Squares. The Cesàro-one sums
\[ \sigma_n(x, t) = \frac{1}{n} \sum_{j=0}^{n-1} K_j(x, t), \]
are expressed in the way suggested by Christoffel's method as follows:
\[ n(x - t)^2\sigma_n(x, t) = \sum_{j=0}^{n-1} \frac{k_j}{k_{j+1}} (b_j - b_{j+1}) [p_{j+1}(x)p_j(t) + p_{j+1}(t)p_j(x)] \]
\[ + \frac{k_{n-1}}{k_n} [p_{n+1}(x)p_{n-1}(t) + p_{n-1}(x)p_{n+1}(t)] \]
\[ - 2 \left( \frac{k_{n-1}}{k_n} \right)^2 p_n(x)p_n(t) + 2 \sum_{j=0}^{n-1} p_j(x)p_j(t) \left\{ \left( \frac{k_j}{k_{j+1}} \right)^2 - \left( \frac{k_{j-1}}{k_j} \right)^2 \right\}, \]
where $b_j = \int_I t[p_j(t)]^2w(t) \, dt$ and $k_{-1} = 0$.

Beginning with $k_2(b_1 - b_0)/k_1 = c_{1,2}$, we see that $b_j = b_{j+1}$ for all $j$ if and only if $w$ is symmetric over $I$. After a translation, we may assume in this case that the $p_i(t)$ are alternately even and odd polynomials. We assume that this condition holds in the sequel.

Let
\[ \Lambda_j(x) = \frac{k_{j-1}}{k_j} p_{j-1}(x) - \frac{k_j}{k_{j+1}} p_{j+1}(x), \]
so that
\[ 4 \frac{k_{j-1}}{k_{j+1}} p_{j+1}(x)p_{j-1}(x) = x^2|p_j(x)|^2 - |\Lambda_j(x)|^2. \]
Then, for suitable constants $c_n$, we set
\[ L_n(x) = (c_n^2 - x^2)|p_n(x)|^2 + |\Lambda_n(x)|^2 \]
\[ = 4 \sum_{j=0}^{n-1} |p_j(x)|^2 \left\{ \left( \frac{k_j}{k_{j+1}} \right)^2 - \left( \frac{k_{j-1}}{k_j} \right)^2 \right\} + \left\{ c_n^2 - 4 \left( \frac{k_{n-1}}{k_n} \right)^2 \right\} |p_n(x)|^2. \]
To make this formulation of sums of squares useful, the weight function $w$ is further restricted.

4. Gegenbauer Polynomials. See [1].
The expansion of $\rho^{-2\lambda} = (1 - 2rt + r^2)^{-\lambda}$ as a power series in $r$,
\[ (1 - rz)^{-\lambda}(1 - rz)^{-\lambda} = \sum_{j=0}^{\infty} C_j^\lambda(t)r^j, \]
subject to
\[ z + \bar{z} = 2t = 2 \cos \theta, \quad z\bar{z} = 1, \quad 0 \leq r < 1, \]
determines the Gegenbauer polynomials \( C_n^\lambda \) of order \( \lambda > 0 \). If \( y \) is any successively differentiable function of \( \rho \),
\[ r^2 \frac{\partial^2 y}{\partial r^2} + \frac{\partial^2 y}{\partial t^2} = r^2 \frac{\partial^2 y}{\partial \rho^2}. \]
In the above case, \( y = \rho^{-2\lambda} \), so \( \frac{d^2 y}{d \rho^2} + ((2\lambda + 1)/\rho)(\frac{dy}{d \rho}) = 0 \), and so
\[ r^2 \frac{\partial^2 y}{\partial r^2} + (2\lambda + 1) \frac{\partial y}{\partial r} + (1 - t^2) \frac{\partial^2 y}{\partial t^2} = (2\lambda + 1) t \frac{\partial y}{\partial t}. \]
Comparing coefficients in the power series, we have
\[ \frac{d}{dt} \left\{ (1 - t^2)^{\lambda+1/2} \frac{dC_n^\lambda(t)}{dt} \right\} = -n(n + 2\lambda)(1 - t^2)^{\lambda-1/2}C_n^\lambda(t). \]
Multiplying by \( C_j^\lambda(t) \), alternating the indices \( n \) and \( j \), and subtracting, then integrating from \( t = -1 \) to \( t = 1 \), we have
\[ C_j^\lambda(t) = \sqrt{h_j} p_j(t), \]
the \( \{p_j\} \) being orthogonal (with property \( 5 \)) with respect to \( w \),
\[ w(t) = (1 - t^2)^{\lambda-1/2}. \]
Here,
\[ \int_{-1}^{+1} |C_j^\lambda(t)|^2 w(t) dt = h_j, \]
easily calculated explicitly. From the definition above, using the series and the binomial theorem,
\[ C_n^\lambda(\cos \theta) = \sum_{j=0}^{n} \binom{\lambda + j - 1}{j} \binom{\lambda + n - j - 1}{n - j} \cos (n - 2j\theta), \]
so
\[ |C_n^\lambda(t)| \leq C_n^\lambda(1) = \binom{2\lambda + n - 1}{n}, \quad -1 \leq t \leq 1, \]
if \( \lambda > 0 \).
We may make direct use of the Christoffel formulae \( 7 \), comparison of terms in a linear expansion, and induction, to obtain
\[ 2h_n k_0^2 (n + \lambda) = \lambda \binom{n + 2\lambda - 1}{n}, \]
\[ 4 \left( \frac{k_{n-1}}{k_n} \right)^2 = \frac{n(n + 2\lambda - 1)}{(n + \lambda)(n - 1 + \lambda)}, \]
and

\[ \lambda^2 2^{2\lambda} (n + \lambda) C_n^\lambda(t)^2 = \pi \left( \frac{2\lambda + n - 1}{n} \right) \lambda(2\lambda)! \{p_n(t)\}^2. \]

Also,

\[ \frac{k_{n-1}}{k_n} p_{n-1}(0) = -\frac{k_n}{k_{n+1}} p_{n+1}(0), \]

so that

\[ \lim_{n \to \infty} \{p_\infty(0)\}^2 = \frac{2}{\pi} \]

and

\[ \lim_{n \to \infty} p_{n+1}(1) (n + \lambda)^{-2\lambda} = \sqrt{\frac{2}{\pi}} \frac{2\lambda!}{(2\lambda)!}, \]

the relative errors in the corresponding approximations being of (order) \( O(1/(n + \lambda)^2) \) uniformly in \( n \) for fixed \( \lambda \) by Stirling's formula.

We set \( \lambda = (1 - t^2) \frac{\lambda}{\lambda} \), and find

\[ \frac{d\lambda}{dt} = (n + \lambda)(1 - t^2) X/2 - 1 A^nW, \]

using (6) and (11). If

\[ L_n(t) = \{p_n(t)\}^2 (1 - t^2) + \{\Lambda_n(t)\}^2, \]

(11) becomes

\[ \frac{d}{dt} \left[ L_n(t)(1 - t^2)^{\lambda-1} \right] = \frac{-2\lambda(1 - \lambda)}{n + \lambda} (1 - t^2)^{\lambda-2} p_n(t) \Lambda_n(t). \]

From the above quadratic relation, and (6),

\[ 2\sqrt{(1 - t^2)} \left| p_n(t) \Lambda_n(t) \right| \leq L_n(t). \]

Differentiating the logarithm of \( L_n \), and integrating, we have

\[ \log \left( \frac{L_n(t)}{L_n(0)} \right) (1 - t^2)^{\lambda-1} < \frac{\lambda(1 - \lambda)}{n + \lambda} \frac{|t|}{\sqrt{(1 - t^2)}}, \quad 0 < |t| < 1. \]

In particular, \( \lim_{n \to \infty} L_n(t)(1 - t^2)^{\lambda-1} = 2/\pi, -1 < t < 1. \)

However, relation (10) now reads as follows:

\[ L_n(t) = -2 \sum_{j=0}^{n-1} \frac{\lambda(1 - \lambda) \{p_j(t)\}^2}{(j - 1 + \lambda)(j + \lambda)(j + 1 + \lambda)} - \frac{\lambda(1 - \lambda) \{p_n(t)\}^2}{(n + \lambda - 1)(n + \lambda)}, \]

whence

\[ L_n(t)(1 - t^2)^{\lambda-1} = \{p_n(t)\}^2 (1 - t^2)^{\lambda} + \{\Lambda_n(t)\}^2 (1 - t^2)^{\lambda-1} \]

\[ = \frac{2}{\pi} \frac{\lambda(1 - \lambda) \{p_n(t)\}^2 (1 - t^2)^{\lambda-1}}{n + \lambda(n + \lambda + 1)} \]

\[ + 2 \sum_{j=n+1}^{\infty} \frac{\lambda(1 - \lambda) \{p_j(t)\}^2 (1 - t^2)^{\lambda-1}}{(j - 1 + \lambda)(j + \lambda)(j + 1 + \lambda)}. \]
The maximum of \( z^2 = \{p_n(t)\}^2(1 - t^2)^\lambda \) in any subinterval of \( I \) with endpoints \( t = x_i \) or \( t = \pm 1 \), corresponds only to \( \Lambda_n(t) = 0 \), so that if
\[
(n + \lambda)(n + \lambda + 1)(1 - t^2) \geq \frac{|\lambda(1 - \lambda)|}{\epsilon},
\]
\[
p_n(t)(1 - t^2)^\lambda(1 \pm \epsilon) < \frac{2}{\epsilon},
\]
and, otherwise,
\[
p_n(1)(1 - t^2)^\lambda
\]
is uniformly bounded, by (12) and (13).

On the other hand, if \( p_n(x_i) = 0 \),
\[
\{\Lambda(x_i)\}^2(1 - x_i^2)^{\lambda-1} = 2 + \frac{2\lambda(1 - \lambda)}{|1 - x_i^2|} \sum_{j=n+1}^{\infty} \frac{|p_j(x_i)|^2(1 - x_i^2)^\lambda}{(j - 1 + \lambda)(j + \lambda)(j + 1 + \lambda)},
\]
where
\[
\sum_{j=n+1}^{\infty} \frac{|p_j(x)|^2(1 - x^2)^\lambda}{(j - 1 + \lambda)(j + \lambda)(j + 1 + \lambda)} < \frac{1 + \epsilon_n}{\pi(n + \lambda)^2}
\]
and \( \lim_{n \to \infty} \epsilon_n = 0 \), if \( |\pm 1 + x| > \delta \), any fixed positive number. That is, if \( |\pm 1 + x_i| > \delta \),
\[
\frac{1}{W_i} = K_n(x_i, x_i) = \frac{\Lambda_n(x_i)p_n'(x_i)}{2} = \frac{n + \lambda}{2} \left( \frac{\Lambda_n(x_i)}{1 - x_i^2} \right)^2,
\]
and for such zeros \( x = x_i \),
\[
W_i \lesssim \frac{\pi}{n + \lambda} (1 - x_i^2)^\lambda,
\]
with a relative-error estimate
\[
\frac{|\lambda(1 - \lambda)|}{(n + \lambda)^2(1 - x_i^2)}
\]
for both upper and lower bounds.

If \( n \) is an odd number, and \( x_i = 0 \), we easily compute
\[
\frac{1}{W_i} = \frac{n + \lambda}{\pi} \left\{ 1 + \frac{\lambda(1 - \lambda)}{2n^2} + \frac{\lambda(1 - \lambda)^2}{n^3} + \cdots \right\},
\]
using Stirling's formula, for the corresponding median weight \( W_i \). The precision of the estimate here is easily controlled; but in the general case the sums of squares seem difficult to handle with precision.

5. Spacing of Zeros. Let \( v = p_n'(t)/p_n(t) \). Using (11), we find
\[
(1 - t^2) \frac{dv}{dt} = (2\lambda + 1)tv - n(n + 2\lambda) - (1 - t^2)v^2.
\]
Combining this with the Christoffel formulae, using induction and the result \( |p_n(t)| \leq p_n(1) \), we have
\[
v \leq \frac{p_n'(1)}{p_n(1)} = \frac{n(n + 2\lambda)}{2\lambda + 1} \quad \text{if} \quad x_n < t \leq 1,
\]
x = x_n being the zero of p_n(t) nearest t = 1. Since (p_n(x_n) - p_n(1))/(x_n - 1) < p_n'(1), we have x_n < 1 - (2\lambda + 1)/(n(n + 2\lambda)).

In general, if we set t = \sin \phi, the equivalent differential relation
\[-\frac{d}{d\phi} \left\{ \arctan \left[ \frac{\Delta_n(t)}{p_n(t) \sqrt{(1 - t^2)}} \right] \right\}
= n + \lambda + \frac{\lambda(1 - \lambda)}{n + \lambda} \frac{|p_n(t)|^2}{L_n(t)}, \quad x_i < t < x_{i+1},

gives us the necessary information concerning the spacing of the zeros. We have
\[\pi = \Delta \arctan \left[ \frac{\Delta_n(t)}{p_n(t) \sqrt{(1 - t^2)}} \right] = (n + \lambda) \Delta \phi_i + \frac{\lambda(1 - \lambda)}{n + \lambda} \int_{\phi_i}^{\phi_{i+1}} \frac{|p_n(t)|^2}{L_n(t)} d\phi,
\]
where x_i = \sin \phi_i and \Delta \phi_i = \phi_{i+1} - \phi_i.

6. Hermite Polynomials. From the defining formulas, we easily obtain
\[\left( \frac{d}{dt} \right)^m \{ C_n^\lambda(t) \} = 2^n \binom{\lambda + m - 1}{m} C_n^{\lambda + m}(t),
\]
by induction on m. Among other results, relations between the tesseral harmonics of Legendre,
\[P_n^{(m)}(t) = (1 - t^2)^{m/2} \left( \frac{d}{dt} \right)^m \{ P_n(t) \},
\]
\[P_n(t) = C_n^\lambda(t) \quad \text{for} \quad \lambda = \frac{1}{2},
\]
and the Gegenbauer polynomials follow. Formally, the trigonometric basis is given by \lambda = 0 and \lambda = 1.

If t^2 = s^2/2\lambda, s being fixed, and \lambda \to \infty, we have
w(t) \to e^{-s^2/2}.

For the bounded n and s,
\[C_n^\lambda(t) \to H_n(s), \quad \text{if} \quad \lambda \to \infty,
\]
the corresponding Hermite polynomial.

Let
\[\frac{d}{dt} \{ H_n(t)e^{-t^2/2} \} = -H_{n+1}(t)e^{-t^2/2}, \quad H_0(t) = 1,
\]
for n = 0, 1, 2, \cdots. Then
\[H_n'(t) = nH_{n-1}(t),
\]
by Leibnitz' rule for successive differentiation. It follows immediately that
\[H_n(x) = \sum_{j<(n+1)/2} \binom{n}{2j} (-1)^j C_j x^{n-2j}
\]
for a single set of coefficients \( \{C_j\} \). Since

\[
tH_n(t) = nH_{n-1}(t) + H_{n+1}(t)
\]

from the pair of relations given above, we have the Christoffel formulae

\[
H_n(x, t) = \sum_{j=0}^{n} \frac{H_j(x)H_j(t)}{j!} = \frac{H_{n+1}(x)H_n(t) - H_n(x)H_{n+1}(t)}{n!(x - t)}
\]

and

\[
H_n(x, x) = \sum_{j=0}^{n} \frac{H_j^2(x)}{j!} = \frac{(n + 1)H_n^2(x) - nH_{n+1}(x)H_{n-1}(x)}{n!}
\]

\[
= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} H_n^2(x, t)e^{-t^2/2} dt.
\]

To arrive at the last result, we make use of

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2 + t^2)/2} dx dt = 2\pi,
\]

or the limits given above. Since

\[
\sqrt{(2\pi)}e^{-x^2/2} = \int_{-\infty}^{\infty} e^{-t^2/2 + ixt} t^n dt
\]

we have

\[
\sqrt{(2\pi)}H_n(x)e^{-x^2/2} = (-i)^n \int_{-\infty}^{\infty} e^{-t^2/2 + ixt} t^n dt.
\]

Let \( z = e^{-t^2/2}H_n(t) \), so that

\[
\frac{dz}{dt} = e^{-t^2/4} \left\{ nH_{n-1}(t) - \frac{t}{2} H_n(t) \right\}
\]

and

\[
\frac{d^2z}{dt^2} = -z \left( n + 1 - \frac{t^2}{4} \right).
\]

Then

\[
(tz)^2 - 4 \left( \frac{dz}{dt} \right)^2 = 4ne^{-t^2/2}H_{n-1}(t)H_{n+1}(t),
\]

so that

\[
e^{-t^2/2} \left\{ \sum_{j=0}^{n-1} \frac{H_j^2(x)}{j!} + \frac{1}{2} \frac{H_n^2(x)}{n!} \right\} = \sum_{j=0}^{n-1} \frac{H_j^2(0)}{j!} + \frac{1}{2} \frac{H_n^2(0)}{n!} - \frac{1}{2} \int_0^x t e^{-t^2/2} \frac{H_n^2(t)}{n!} dt
\]

from the Christoffel formula. We do not obtain different results from the formulation of the Cesàro-one sums, in this case. We define

\[
L_n(x) = \frac{1}{\sqrt{n}} \left\{ \sum_{j=0}^{n-1} \frac{H_j^2(x)}{j!} + \frac{1}{2} \frac{H_n^2(x)}{n!} \right\},
\]
so that
\[ \lim_{n \to \infty} L_n(0) = \sqrt{\frac{2}{\pi}}. \]
Then, also,
\[ L_n(x)e^{-x^2/2} = L_n(0) - \frac{1}{\sqrt{n}} \int_0^x t \frac{H_n^2(t)}{n!} e^{-t^2/2} \, dt, \]
so here
\[ \sqrt{n} L_n(t)e^{-t^2/2} = \frac{1}{n!} \left( \frac{dz}{dt} \right)^2 + \left( n + \frac{1}{2} - \frac{t^2}{4} \right) z^2, \]
and
\[ \lim_{n} L_n(x)e^{-x^2/2} = \sqrt{\frac{2}{\pi}}. \]
The formula for the weights \( W_i \) corresponding to \( H_n(x_i) = 0 \) becomes
\[ \frac{1}{W_i} = H_n(x_i, x_i) = \sqrt{n} L_n(x_i), \]
so
\[ W_i \approx \sqrt{\frac{\pi}{2n}} e^{-x_i^2/2}, \]
with a relative error estimate
\[ \frac{x_i^2}{2n - \delta} \quad \text{if} \quad x_i^2 < 2(1 + \delta). \]

If we consider the Fourier sine expansion over the interval \((a, a + \pi/k)\) between zeros \( x = a, x = b = a + \pi/k, \) of \( H_n(x)e^{-x^2/4}, \) we have
\[ \int_a^b \left( \frac{dz}{dt} \right)^2 - k^2 z^2 \, dt > 0. \]
Now
\[ \int_a^b \left( \frac{dz}{dt} \right)^2 - \left( n + \frac{1}{2} - \frac{t^2}{4} \right) z^2 \, dt = 0, \]
so that
\[ b - a > \frac{2\pi}{\sqrt{4n + 2 - a^2}}. \]
Otherwise, \( dz/dt < 0 \) if \( t^2 \geq 4n + 2. \) We cannot have \( z = 0 \) there, since \( z > 0 \) if \( t \to \infty \) for fixed \( n. \) Then
\[ b^2 < 4n + 2. \]
We may point out that the estimates, for Cesàro-one and related sums, remain useful in establishing convergence properties of the expansions of functions (e.g., of bounded variation) as series of orthogonal polynomials.

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