On a Constant in the Theory of Trigonometric Series

By Robert F. Church

The note "A constant in the theory of trigonometric series" in the October 1964 issue of Mathematics of Computation provided us with a test for our recently constructed algorithms for the computation of roots of functions, and for numerical quadrature in the presence of singularities. The latter algorithm, utilizing the Gaussian 8-point quadrature formula applied to sub-intervals of variable length, involves a sufficiently small number of ordinates that computational labor and round-off error do not become problems. Use of these algorithms indicated the value \( \alpha_0 = 0.3084438 \), for the root of the equation \( \int_0^{3/2} u^{-\alpha} \cos u \, du = 0 \), differing from the reported value, 0.30483, in the third place. To check this result, we made the transformation \( u = x^4 \) to weaken the character of the singularity at the origin, and obtained the following table by conventional numerical quadrature, confirming our result:

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( F(\alpha) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.308441</td>
<td>(-99 \times 10^{-5})</td>
</tr>
<tr>
<td>0.308442</td>
<td>(-63 \times 10^{-5})</td>
</tr>
<tr>
<td>0.308443</td>
<td>(-28 \times 10^{-5})</td>
</tr>
<tr>
<td>0.308444</td>
<td>(0.8 \times 10^{-5})</td>
</tr>
<tr>
<td>0.308445</td>
<td>(0.44 \times 10^{-5})</td>
</tr>
<tr>
<td>0.308446</td>
<td>(0.79 \times 10^{-5})</td>
</tr>
</tbody>
</table>

Sperry Rand Research Center
North Road
Sudbury, Massachusetts

Received October 28, 1964.

On a Constant in the Theory of Trigonometric Series

By Yudell L. Luke, Wyman Fair, Geraldine Coombs and Rosemary Moran

In a recent note, Boas and Klema [1] considered

\[
(1) \quad F(\alpha) = \int_0^{3/2} u^{-\alpha} \cos u \, du, \quad R(\alpha) < 1,
\]

and gave some computations from which they concluded that a zero \( \alpha_0 \) of \( F(\alpha) \) lies between 0.30483 and 0.30484. Since their tabulated values of \( F(\alpha) \) in the vicinity of the root are given to 8D and there are eight such entries, it would seem, since \( F(\alpha) \) is analytic for \( R(\alpha) < 1 \), that the zero could be given to more places by differencing and making use of ordinary inverse interpolation techniques. It is found

Received December 15, 1964.
that the differences are not smooth.* This has led us to determine the zero anew. We find that, to 15D,
\[ \alpha_0 = 0.308443779561985. \]
Thus the value in [1] is incorrect in the third place. The computation was done in two different ways using the main diagonal Padé approximations for the incomplete gamma functions \( \gamma(n, z) \) and \( \Gamma(n, z) \), see [2, 3, 4]. Thus, with
\[
(2) \quad \gamma(n, z) = \int_0^t e^{-t} t^{-1} dt, \quad \Gamma(n, z) = \int_z^{e^{-z}} e^{-t} t^{-1} dt = \Gamma(n) - \gamma(n, z),
\]
we have
\[
(3) \quad |\theta| < \pi/2, \quad R(n) > 0; \quad |\theta| = \pi/2, \quad 0 < R(n) < 1,
\]
and
\[
(4) \quad F(\alpha) = R\{e^{-ir/2} \gamma(n, z^{r/2})\}, \quad \nu = 1 - \alpha, \quad z = 3\pi/2.
\]
For the evaluation of \( \Gamma(n) \), we used an (unpublished) expansion in series of Chebyshev polynomials of the first kind. The basic theory for its development can be found in [5]. All calculations were done on an IBM 1620 computer. After locating the value of \( \alpha_0 \) to about 8D, we evaluated \( F(\alpha) \) for \( \alpha = 0.30844380 \pm nh, n = 0(1)3, h = 0.5 \cdot 10^{-7}. 20D were carried and the truncation error assured an accuracy of about 18D. The values of \( F(\alpha) \) were differenced and \( \alpha_0 \) was found by inverse interpolation using the approach outlined in [6]. In this computation third, and higher differences were ignored as the second differences are essentially constant to 18D, and, rounded to 16D, they are \( 0.105 \cdot 10^{-13} \).

Midwest Research Institute
Kansas City, Missouri


* We are indebted to Dr. John W. Wrench, Jr., for this information.