and take

\[ X_0 = \frac{1}{2} A^* = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}. \]

Here, formula (17) is used to obtain:

\[ X_1 = \frac{1}{2} \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ -1 & 1 \end{pmatrix}, \]

\[ X_2 = \frac{1}{16} \begin{pmatrix} 10 & 5 \\ 5 & 10 \\ -5 & 5 \end{pmatrix}, \]

\[ X_3 = \frac{1}{256} \begin{pmatrix} 170 & 85 \\ 85 & 170 \end{pmatrix}, \quad \text{etc.,} \]

converging to:

\[ A^+ = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ -1 & 1 \end{pmatrix}. \]

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A Note on the Maximum Value of Determinants over the Complex Field

By C. H. Yang

The purpose of this note is to extend a theorem on determinants over the real field to the corresponding theorem over the complex field.

Theorem. Let \( D(n) \) be an \( n \)th order determinant with complex numbers as its entries. Then

\[ \text{Max } |D(n)| = \text{Max } |D(n)|. \]

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In other words, \( D(n) \) is a function of \( n^2 \) variables \( a_{jk} \) which vary over the bounded and closed domain \( \bar{D} : \{ |a_{jk}| \leq K \} \); hence this function is bounded and attains its maximum value on the boundary of the domain \( \bar{D} \).

**Proof.** Let \( a_{jk} = r_{jk} e^{i\theta_{jk}} \) and \( A_{jk} = R_{jk} e^{i\phi_{jk}} \) be the co-factor of \( a_{jk} \), where \( K \geq r_{jk} \geq 0 \) and \( R_{jk} \geq 0 \). Then, expanding by the \( j \)th row, we have

\[
|D(n)| = \left| \sum_{k=1}^{n} a_{jk} A_{jk} \right| = \left| \sum_{k=1}^{n} r_{jk} R_{jk} e^{i(\theta_{jk} + \phi_{jk})} \right|
\]

\[
\leq \sum_{k=1}^{n} r_{jk} R_{jk} \leq \sum_{k=1}^{n} KR_{jk} = D'(n),
\]

where \( D'(n) \) is the \( n \)th order determinant whose entries are

\[
a'_{jk} = \begin{cases} 
    a_{jk}, & \text{if } r_{jk} = K \text{ and } \theta_{jk} + \phi_{jk} = 0 \pmod{2\pi}, \\
    Ke^{-i\phi_{jk}}, & \text{if } r_{jk} < K \text{ or } \theta_{jk} + \phi_{jk} \neq 0 \pmod{2\pi}.
\end{cases}
\]

By applying the same process to the other rows, we obtain a determinant \( D^*(n) \) whose entries \( |a'_{jk}| = K \) and \( |D^*(n)| \geq |D(n)| \). Hence, \( \text{Max}_{|a_{jk}| \leq K} |D(n)| \leq \text{Max}_{|a_{jk}| \leq K} |D(n)| \); thus the proof of the theorem can be completed since the reverse inequality is trivial.

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**On the Numerical Solution of** \( y' = f(x, y) \) **by a Class of Formulae Based on Rational Approximation**

By John D. Lambert and Brian Shaw

1. **Introduction.** Most finite difference formulae in common usage for the numerical solution of first-order differential equations are based on polynomial approximation. Two exceptions are the formulae based on exponential approximation proposed by Brock and Murray [1], and the formulae of Gautschi [2] which are derived from trigonometric polynomials. The use of rational functions as approximants has been studied by many authors, including Remes [3], Maehly [4] and Stoer [5], but the main concern of most of this work has been the direct approximation of a given function. Algorithms for interpolation based on rational functions have been proposed by Wynn [6], and methods for numerical integration and differentiation based on Padé approximation have been studied by Kopal [7]. It is the purpose of the present paper to derive a class of formulae, based on rational approximation, for the numerical solution of the initial value problem

\[
y' = f(x, y), \quad y(x_0) = y_0.
\]

The formulae proposed give exact results when the theoretical solution of (1) is a rational function of a certain degree, just as many of the classical difference formulae

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