On the Sum of the Series \( \sum_{i=0}^{\infty} \left( t^i / [uv + m] \right) \)

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An estimation of error in summing the series

\[
\sum_{i=0}^{\infty} \frac{t^i}{\Gamma(\beta \nu + 2)}, \quad 0 < \beta < 1,
\]

obtained by a solution of Volterra’s equation

\[
\phi(x) = ax + b \int_0^x (x - z)^{\beta - 1} \phi(z) \, dz, \quad a = \text{const}, \, b = \text{const},
\]

leads to series of the form

\[
S_{u,m}(t) = \sum_{i=0}^{\infty} \frac{t^i}{[uv + m]!},
\]

where \([x]\) denotes the largest integer no greater than the real number \(x\), \(m\) a non-negative integer, and \(u\) real, positive and rational.

The series (3) is absolutely and uniformly convergent for all real values of \(t\) and therefore not dependent on the sequence of terms. Changing the sequence of terms in (3) we obtain

\[
S_{u,m}(t) = \sum_{i=0}^{q-1} \sigma^{i}_{u,m}(t),
\]

where

\[
\sigma^{i}_{u,m}(t) = \sum_{i=0}^{\infty} \frac{t^{i+i}}{[uj + m + pi]!}
\]

and

\[
u = \frac{p}{q},
\]

where \(p\) and \(q\) are positive integers with greatest common divisor equal to 1. Substituting in (5)

\[
z^p = t^q,
\]

we have

\[
\sigma^{i}_{u,m}(t) = t^{i-[uj+m]/u} \sigma^{i}_{u,m}(z),
\]

where

\[
s^{i}_{u,m}(z) = \sum_{i=0}^{\infty} \frac{z^{[uj+m+pi]}}{[uj + m + pi]!}.
\]

Differentiating (9) \([uj + m]\) times with respect to \(z\) we obtain

\[
s(z) = \frac{d^{[uj+m]} \sigma^{i}_{u,m}(z)}{dz^{[uj+m]}} = \sum_{i=0}^{\infty} \frac{z^{pi}}{(pi)!}.
\]

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* \(z\) is an imaginary number when \(q\) is odd and \(p\) even.
But (10) is the solution of the pth order linear differential equation

\[(11) \quad s^{(p)}(z) = s(z)\]

with initial conditions

\[(12) \quad s(0) = 1, \quad s^{(k)}(0) = 0 \quad (k = 1, 2, \ldots, p - 1).\]

The characteristic equation of (12) is

\[(13) \quad r^p - 1 = 0.\]

If \(r\) denotes a primitive root of pth order of unity, the general solution of (11) has the form

\[(14) \quad s(z) = \sum_{i=1}^{p} A_i \exp r^i z,\]

and taking (12) into consideration we have

\[(15) \quad \sum_{i=1}^{p} A_i = 1, \quad A_i = \text{const},\]

\[(16) \quad \sum_{i=1}^{p} A_i r^k = 0 \quad (k = 1, 2, \ldots, p - 1).\]

For finding \(A_i\) we use Cramer's rule, and since there are Vandermonde's determinants in our case

\[(16) \quad A_i = \frac{\prod_{k=2}^{p} \prod_{n=1}^{k-1} (b_k - b_n)}{\prod_{k=2}^{p} \prod_{n=1}^{k-1} (r^k - r^n)}, \quad \text{where} \quad b_n = \begin{cases} r^n & \text{when} \ n \neq l, \\ 0 & \text{when} \ n = l, \end{cases}\]

or

\[(17) \quad A_i = \prod_{k=1, k \neq i}^{p} \frac{1}{1 - r^{-i-k}} = \prod_{k=1}^{p-1} \frac{1}{1 - r^k},\]

it follows that all \(A_i\) are equal, and, thus, from (15),

\[(18) \quad A_i = \frac{1}{p}.\]

Putting (18) into (14) we have

\[(19) \quad s(z) = \frac{1}{p} \sum_{i=1}^{p} \exp (r^i z).\]

Integrating both sides of (19) \([u_j + m]\) times in the interval \((0, z)\) we obtain

\[(20) \quad s_{u, m}^j(z) = \frac{1}{p} \sum_{i=1}^{p} \exp (r^i z) r^{-i[u_j + m]} - B,\]

where

\[(21) \quad B = \frac{1}{p} \sum_{k=1}^{[u_j + m]} \frac{z^{[u_j + m] - k}}{[u_j + m - k]!} \sum_{i=1}^{p} r^{-i k}.\]
But

\[ \sum_{i=1}^{p} r^{-lk} = \sum_{i=1}^{p} r^{lk} = \begin{cases} 0 & \text{when } k \text{ is divisible by } p, \\ p & \text{when } k \text{ is not divisible by } p, \end{cases} \]

and therefore

\[ B = \sum_{k=1}^{[\frac{(uj+m)}{p}]} \frac{z^{[uj+m]-pk}}{[uj+m-pk]!}. \]

Returning to the variable \( t \) we obtain finally

\[ S_{u,m}(t) = \sum_{j=0}^{q-1} t^{\{-uj+m\}/u} \]

\[ \cdot \left\{ \frac{1}{p} \sum_{i=1}^{p} \exp \left( r^{i} t^{u} \right) r^{-[uj+m]} \sum_{k=1}^{[\frac{(uj+m)}{p}]} \frac{t^{[uj+m-pk]/u}}{[uj+m-pk]!} \right\}. \]

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