A Division Algebra for Sequences Defined on all the Integers

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The convolution ring, S, of sequences defined on the nonnegative integers, and the embedding of this ring in a field, have been discussed by Brand [1], Moore [2], [3], Traub [6], and others. Brand [1] specifically mentions that the field in which he embeds S is a field of ordered pairs of members of S. Traub does not identify his field and does not mention "ordered pairs", but he mentions an analogy to Mikusiński's work [7], and so he probably had in mind the same field of ordered pairs as did Brand. In [2] this writer showed that it was not necessary to create such a field of ordered pairs since there already existed a more natural, less abstract field in which to embed S. It is the purpose of this article to introduce this already existing and more natural field, F, in which S may be embedded.

It will be assumed that the reader is familiar with the convolution algebra of sequences as given in [1], [3], and [6] to the point of recognizing S as an integral domain in which convolution products defined by

\[ \{a_r\} \{b_r\} = \left( \sum_{\mu=0}^{r} a_\mu b_{r-\mu} \right) \]

contain no divisors of zero, in which the multiplicative unity is the sequence

\[ \{1, 0, 0, 0, \cdots, 0, \cdots\}, \]

in which sequences of the form

\[ \{c, 0, 0, 0, \cdots, 0, \cdots\} \]

behave like numbers and are identified with numbers:

\[ c = \{c, 0, 0, 0, \cdots, 0, \cdots\}, \]

in which the sequence

\[ \{0, 1, 0, 0, 0, \cdots, 0, \cdots\} \]

is a shift operator denoted by "\( \tau \)", in which the sequence

\[ \{1, 1, 1, \cdots, 1, \cdots\} \]

is a summing operator denoted by "\( \sigma \)”, and in which members of S have operational forms in terms of \( \tau \) and/or \( \sigma \).

The sequences \( \sigma \) and \( \tau \) are related by the equation

\[ \sigma(1 - \tau) = 1 \]

and since S has no divisors of zero we introduce fractions and write (for example)

\[ \sigma = \frac{1}{1 - \tau}, \]

\[ \frac{1}{\sigma} = 1 - \tau = \{1, -1, 0, 0, 0, \cdots\}. \]
The fraction $1/\tau$, for example, does not exist as a member of $\mathcal{S}$. But $1/\tau$ will exist as a member of $\mathcal{F}$.

Let $\mathcal{F}$ be the class of number valued sequences defined over the integral domain, $J$, each of which assigns at most a finite number of nonzero values to negative integers. For each member of $\mathcal{F}$ there is a least integer, $m$, to which the sequence assigns a nonzero value; the sequence will be said to enter at $m$, and the members of $\mathcal{F}$ will be called entering sequences. Equality, sums, and products with numbers, of members of $\mathcal{F}$ are defined in the usual termwise way. A suggested notation for such a sequence is

\[
\{1, 2, 3 \mid 4, 5, 6, \ldots\}
\]

where the vertical line—playing a role like a decimal point—separates values assigned to the negative integers from values assigned to the nonnegative integers, and the zeros assigned on the left are omitted for convenience.

Let $v$ be a variable on $J$. We define the unit step formula $u(v)$:

\[
u < 0, \quad u(v) = \begin{cases} 0, \\ 1, \quad v \geq 0. \end{cases}
\]

Then $\{u(v)\}$ and $\{(v + 1)u(v)\}$ (for example) are members of $\mathcal{F}$, whereas $\{v + 1\}$ is not. The braces serve to bind out "$v$" converting a formula into a notation for a sequence.

There is a natural one-to-one correspondence between $\mathcal{S}$ (sequences defined on the nonnegative integers) and the subclass, $\mathcal{F}_0$, of $\mathcal{F}$ consisting of sequences which enter at nonnegative points:

\[
\{a_0, a_1, a_2, \ldots\} \leftrightarrow \{\ldots, 0, 0 \mid a_0, a_1, a_2, \ldots\}.
\]

The convolution, or convolution product, of two sequences $a$ and $b$ from $\mathcal{F}$ is defined by

\[
ab = \left\{ \sum_{\mu=-\infty}^{+\infty} a_\mu b_{\nu-\mu} \right\}. \tag{3}
\]

If $c$ enters at $\alpha$ or to the right of $\alpha$ and $b$ enters at $\beta$ or to the right of $\beta$, then

\[
(ab)_v = \begin{cases} \sum_{\mu=\alpha}^{\nu-\beta} a_\mu b_{\nu-\mu}, & \nu \geq \alpha + \beta, \\ 0, & \nu < \alpha + \beta, \end{cases} \tag{4}
\]

\[
ab = \left\{ u(\nu - \alpha - \beta) \sum_{\mu=\alpha}^{\nu-\beta} a_\mu b_{\nu-\mu} \right\}. \tag{5}
\]

The summation limits are finite in (5) since the sequences are entering sequences. In particular, if $a$ and $b$ are members of the subclass $\mathcal{F}_0$, we may take $\alpha = \beta = 0$ in (5) and (5) becomes

\[
ab = \left\{ u(\nu) \sum_{\mu=0}^{\nu} a_\mu b_{\nu-\mu} \right\}. \tag{6}
\]

A comparison of (1) and (6) shows that the correspondence (2) is an isomorphism under convolution; we embed $\mathcal{S}$ in $\mathcal{F}$, identify $\mathcal{S}$ with $\mathcal{F}_0$, elevate (2) to an
equality, and permit any notation for a member of \( S \) to be used as a notation for the corresponding member of \( \mathcal{S}_0 \). In particular

\[
1 = \{\ldots, 0, 0, 0 | 1, 0, 0, 0, \ldots, 0, \ldots\}
\]

\[
\sigma = \{\ldots, 0, 0, 0 | 1, 1, \ldots, 1, \ldots\}
\]

\[
\tau = \{\ldots, 0, 0, 0 | 0, 1, 0, 0, 0, \ldots, 0, \ldots\}
\]

\[
\tau^n = \{\ldots, 0, 0, 0 | 0, 0, \ldots, 0, 1, 0, 0, 0, \ldots\}
\]

\( m \) zeros

Defining \( \xi \) by:

\[
\xi = \{1 | 0, 0, 0, \ldots, 0, \ldots\}
\]

we have

\[
\xi^m = \{1, 0, 0, 0, \ldots, 0 | 0, 0, 0, \ldots, 0, \ldots\} \quad m \text{ positive integer.}
\]

Equations (7) and (9) may be verified by induction. Using (5) we may verify that

\[
\tau^n = 1
\]

\[
\tau^{m+n} = \xi^m \tau^n
\]

\[
\xi^{m+n} = \xi^m \xi^n
\]

\( m, n \) positive integers.

\[
\tau^m[a_r] = \{a_{r-m}\}
\]

\[
\xi^m[a_r] = \{a_{r+m}\}
\]

Under ordinary addition and convolution multiplication \( \mathcal{S} \) is a field. We need only verify here that each nonzero member of \( \mathcal{S} \) has a multiplicative inverse. To begin with, every sequence of the form

\[
\{a_0, a_1, a_2, \ldots\}
\]

in which \( a_0 \neq 0 \) (the sequence enters at the origin) has an inverse:

\[
\{x_0, x_1, x_2, \ldots\}
\]

which may be evaluated as follows:

\[
\{a_0, a_1, a_2, \ldots\} \{x_0, x_1, x_2, \ldots\} = \{1, 0, 0, 0, \ldots\}
\]

\[
a_0 x_0 = 1
\]

\[
a_0 x_1 + a_1 x_0 = 0
\]

\[
a_0 x_2 + a_1 x_1 + a_2 x_0 = 0
\]

\[
\vdots
\]

Since the only division involved in solving for the \( x \)'s is division by \( a_0 \), and \( a_0 \neq 0 \), the \( x \)'s exist and so the desired inverse exists.

Finally, let \( a \) be any nonzero member of \( \mathcal{S} \) which does not enter at the origin.
Since $a$ is an entering sequence, there exists a sequence $A$ and a positive integer $m$ such that either

\[(10) \quad a = r^mA \quad \text{or} \quad a = \tau^mA\]

where $A$ enters at the origin, and so has an inverse $A^{-1}$ by the preceding paragraph. Then either

\[(A^{-1}r^m)a = 1 \quad \text{or} \quad (A^{-1}\tau^m)a = 1\]

and so, in any case, $a$ has a multiplicative inverse, and $\mathcal{F}$ is a field.

Since $\mathcal{F}$ contains no divisors of zero, products lead to the introduction of fractions:

\[
\begin{align*}
 a, b, c &\in \mathcal{F} \\
 \text{and} \\
 ab &= c \\
 \text{and} \\
 a \neq 0 \\
\end{align*}
\]

\[
\Rightarrow \quad \begin{cases} 
 c &\exists \text{ as a member of } \mathcal{F} \\
 \frac{c}{a} &\text{and} \\
 c &= b \\
 \frac{c}{a} &= b \\
 a \left( \frac{c}{a} \right) &= c.
\end{cases}
\]

In particular

\[
\zeta = \left\{ 1 \mid 0, 0, 0, \cdots \right\} = \frac{1}{\tau} = \frac{\{ 1, 0, 0, 0, \cdots \}}{\{ 0, 1, 0, 0, 0, \cdots \}}
\]

and $1/\tau$ exists as a member of $\mathcal{F}$.

Members of $\mathcal{F}$ may be put into operational form in terms of $\sigma, \tau,$ and/or $\zeta$.

**Example 1.**

\[
\left\{ \frac{\nu(-1)}{2} u(\nu + 2) \right\} = \{ 3, 1 \mid 0, 0, 1, 3, 6, 10, 15, \cdots \}
\]

\[
= \zeta^2 \left\{ \left( \frac{\nu - 2}{2} \right) \left( \frac{\nu - 3}{2} \right) u(\nu) \right\} = \frac{\zeta^2}{2} \left\{ (\nu^2 - 5\nu + 6)u(\nu) \right\}
\]

\[
= \frac{\zeta^2}{2} (\sigma^2 \tau + 2\sigma^2 \tau^2 - 5\sigma^2 \tau + 6\sigma)
\]

where $\{\nu u(\nu)\} = \sigma^2 \tau$ and $\{\nu^2 u(\nu)\} = \sigma^2 \tau + 2\sigma^3 \tau^2$ as shown in [2], and as may be checked straightforwardly. Then

\[
\left\{ \frac{\nu(-1)}{2} u(\nu + 2) \right\} = \sigma^3 - 2\sigma^2 \zeta + 3\sigma^2 \zeta^2.
\]

In Traub [6, p. 196], every quotient of “generalized” sequences with a nonzero denominator equals a shift operator times an ordinary sequence. Thus, in Traub’s notation,

\[
\frac{f}{g} = \frac{f}{\omega e} = \omega^{-1} \frac{f}{g'}
\]
where \( f/e \) equals an ordinary sequence since \( e \) assigns a nonzero value to the origin; \( \omega^{-i} \) is a shift operator, and is a "generalized" sequence—an ordered pair of ordinary sequences. In comparison, in the present paper, we are dealing with entering sequences (defined on \( J \)) instead of ordered pairs, and every quotient, \( b/a \), of entering sequences (with nonzero denominator) equals an entering sequence. In evaluating \( b/a \) we may replace \( a \), as in (10), by \( \tau^m A \) or \( \zeta^m A \), as appropriate, and obtain respectively

\[
\frac{b}{a} = \zeta^m \frac{b}{\bar{A}} \quad \text{or} \quad \frac{b}{a} = \tau^m \frac{b}{\bar{A}}
\]

where \( b/A, \zeta^m \), and \( \tau^m \) are all entering sequences.

**Example 2.**

\[
\begin{align*}
\left\{1, -1, 1, -1, 1, -1, \cdots\right\} & = \frac{1}{1+(1+r)} \cdot 1 - r \\
\left\{1, 1, 1 | 1, 1, \cdots\right\} & \frac{1}{1+r} \cdot \frac{1}{1+r} \quad \text{(see example 1)}
\end{align*}
\]

\[
\left\{1, -2, 2, -2, 2, -2, \cdots\right\} = \left\{0, 0, 0, 1, -2, 2, -2, 2, -2, \cdots\right\}
\]

**Example 3.**

\[
\begin{align*}
\left\{3, 1 | 0, 0, 1, 3, 6, 10, 15, \cdots\right\} & = \frac{\sigma^3 - 2\xi^2 + 3\xi^3}{(1-\tau^3)(1-r^2)^3} \\
\left\{0, 0, 0, 1, 3, 3, 1, 0, 0, 0, \cdots\right\} & = (6\xi^3 - 8\xi^4 + 3\xi^5) \left\{1, 0, 3, 0, 6, 0, 10, 0, 15, 0, \cdots\right\}
\end{align*}
\]

\[
\left\{6, 0, 18 | 0, 36, 0, 60, \cdots\right\}
\]

\[
\left\{-8, 0, -24, 0 | -48, 0, -80, 0, \cdots\right\}
\]

\[
\left\{3, 0, 9, 0, 18 | 0, 30, 0, 45, \cdots\right\}
\]

\[
\left\{3, -8, 15, -24, 36 | -48, 66, -80, 105, \cdots\right\}
\]

The last result may be checked by cross multiplication:

\[
\left\{0, 0, 0, 1, 3, 3, 1, 0, 0, 0, \cdots\right\}\left\{3, -8, 15, -24, 36 | -48, 66, -80, 105, \cdots\right\}
\]

\[
\left\{3, 1 | 0, 0, 1, 3, 6, 10, 15, \cdots\right\}
\]

A convenient way to multiply two entering sequences is to ignore the vertical lines at first, and then insert a vertical line in the final answer, following rules similar to those for the insertion of a decimal point in a product of decimals.
George Boole’s operator, $E$, [4, p. 16] which shifts a sequence to the left and replaces by zero the terms which pass the origin, operates only on sequences which vanish to the left of the origin:

$$E^n[f(\nu)u(\nu)] = [f(\nu + n)u(\nu)], \quad n = \text{nonnegative integer}.$$ 

Thus $E$ cannot be identified with $\zeta$; neither is $E$ to be discarded, since there is no convolution product to do the job that $E$ does, and that job is important. However, George Boole’s symbolic method [4, p. 215] is salvaged if $E$ is replaced by $\zeta$ as discussed in [2]. Thus, Boole’s symbolic equation [4, pp. 217, 218]

$$bx - a \quad b - a \quad c = \text{arbitrary constant} \quad a, \quad b \quad \text{numbers}$$

becomes:

$$\left\{ \frac{b' u(\nu)}{\zeta - a} \right\} = \frac{b' u(\nu)}{b - a} + \frac{a' u(\nu)}{a - b}.$$ 

This follows from the equation

$$\left\{ c' u(\nu) \right\} = \frac{1}{1 - c} \cdot \frac{\zeta}{\zeta - c}, \quad c = \text{number}$$

which is easily checked by cross multiplication. To prove (11) we have

$$\left\{ \frac{b' u(\nu)}{\zeta - a} \right\} = \frac{\zeta}{\zeta - b} \cdot \frac{1}{\zeta - a} = \frac{1}{b - a} \cdot \frac{\zeta}{\zeta - b} + \frac{1}{a - b} \cdot \frac{\zeta}{\zeta - a}$$

$$= \frac{b' u(\nu)}{b - a} + \frac{a' u(\nu)}{a - b}.$$ 

When operational forms of sequences are expressed in terms of $\zeta$ they match the $Z$-transforms of sequences as used, for example, by Aseltine [5] (hence the use of “$\zeta$” for the reciprocal of $\tau$). For example [5, p. 259]

$$\left\{ u(\nu) \right\} = \sigma = \frac{1}{1 - \tau} = \frac{\zeta}{\zeta - 1}.$$ 

But now $\zeta$ is a sequence and not a variable, a formula in $\zeta$ equals a sequence rather than being a “transform” of it, and the introduction of the $\zeta$-forms requires no theory of convergence of power series. In [2, pp. 140-143] it is shown that results previously obtained using the theory of functions of a complex variable including branch cuts and the theory of residues, may be obtained by purely algebraic methods from the field properties of $\zeta$.

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