A Division Algebra for Sequences Defined on *all* the Integers

By D. H. Moore

The convolution ring, $S$, of sequences defined on the nonnegative integers, and the embedding of this ring in a field, have been discussed by Brand [1], Moore [2], [3], Traub [6], and others. Brand [1] specifically mentions that the field in which he embeds $S$ is a field of ordered pairs of members of $S$. Traub does not identify his field and does not mention "ordered pairs", but he mentions an analogy to Mikusiński's work [7], and so he probably had in mind the same field of ordered pairs as did Brand. In [2] this writer showed that it was not necessary to create such a field of ordered pairs since there already existed a more natural, less abstract field in which to embed $S$. It is the purpose of this article to introduce this already existing and more natural field, $\mathfrak{F}$, in which $S$ may be embedded.

It will be assumed that the reader is familiar with the convolution algebra of sequences as given in [1], [3], and [6] to the point of recognizing $S$ as an integral domain in which convolution products defined by

\[
\{a_{r}\} \{b_{r}\} = \left\{ \sum_{\mu=0}^{r} a_{\mu} b_{r-\mu} \right\}
\]

contain no divisors of zero, in which the multiplicative unity is the sequence

\[
\{1, 0, 0, 0, \cdots, 0, \cdots\},
\]

in which sequences of the form

\[
\{c, 0, 0, 0, \cdots, 0, \cdots\}
\]

behave like numbers and are identified with numbers:

\[
c = \{c, 0, 0, 0, \cdots, 0, \cdots\},
\]

in which the sequence

\[
\{0, 1, 0, 0, 0, \cdots, 0, \cdots\}
\]

is a shift operator denoted by "$r$", in which the sequence

\[
\{1, 1, 1, \cdots, 1, \cdots\}
\]

is a summing operator denoted by "$\sigma$", and in which members of $S$ have operational forms in terms of $\tau$ and/or $\sigma$.

The sequences $\sigma$ and $\tau$ are related by the equation

\[
\sigma(1 - \tau) = 1
\]

and since $S$ has no divisors of zero we introduce fractions and write (for example)

\[
\sigma = \frac{1}{1 - \tau},
\]

\[
\frac{1}{\sigma} = 1 - \tau = \{1, -1, 0, 0, 0, \cdots\}.
\]


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The fraction \(1/\tau\), for example, does not exist as a member of \(S\). But \(1/\tau\) will exist as a member of \(\mathcal{F}\).

Let \(\mathcal{F}\) be the class of number valued sequences defined over the integral domain, \(J\), each of which assigns at most a finite number of nonzero values to negative integers. For each member of \(\mathcal{F}\) there is a least integer, \(m\), to which the sequence assigns a nonzero value; the sequence will be said to enter at \(m\), and the members of \(\mathcal{F}\) will be called entering sequences. Equality, sums, and products with numbers, of members of \(\mathcal{F}\) are defined in the usual termwise way. A suggested notation for such a sequence is

\[
\{1, 2, 3 \mid 4, 5, 6, \ldots\}
\]

where the vertical line—playing a role like a decimal point—separates values assigned to the negative integers from values assigned to the nonnegative integers, and the zeros assigned on the left are omitted for convenience.

Let \(\nu\) be a variable on \(J\). We define the unit step formula \(u(\nu)\):

\[
u < 0, \quad u(\nu) = \begin{cases} 0, & \nu < 0, \\ 1, & \nu \geq 0. \end{cases}
\]

Then \(\{u(\nu)\}\) and \(\{(\nu + 1)u(\nu)\}\) (for example) are members of \(\mathcal{F}\), whereas \(\{\nu + 1\}\) is not. The braces serve to bind out \(\nu\) converting a formula into a notation for a sequence.

There is a natural one-to-one correspondence between \(S\) (sequences defined on the nonnegative integers) and the subclass, \(\mathcal{F}_0\), of \(\mathcal{F}\) consisting of sequences which enter at nonnegative points:

\[
(a_0, a_1, a_2, \ldots) \leftrightarrow \cdots, 0 | a_0, a_1, a_2, \ldots
\]

The convolution, or convolution product, of two sequences \(a\) and \(b\) from \(\mathcal{F}\) is defined by

\[
ab = \left\{ \sum_{\mu=-\infty}^{+\infty} a_\mu b_{\nu-\mu} \right\}.
\]

If \(c\) enters at \(\alpha\) or to the right of \(\alpha\) and \(b\) enters at \(\beta\) or to the right of \(\beta\), then

\[
(ab)_\nu = \begin{cases} \sum_{\mu=-\infty}^{\nu-\beta} a_\mu b_{\nu-\mu}, & \nu \geq \alpha + \beta, \\ 0, & \nu < \alpha + \beta, \end{cases}
\]

\[
ab = \left\{ u(\nu - \alpha - \beta) \sum_{\mu=\alpha}^{\nu-\beta} a_\mu b_{\nu-\mu} \right\}.
\]

The summation limits are finite in (5) since the sequences are entering sequences. In particular, if \(a\) and \(b\) are members of the subclass \(\mathcal{F}_0\), we may take \(\alpha = \beta = 0\) in (5) and (5) becomes

\[
ab = \left\{ u(\nu) \sum_{\mu=0}^{\nu} a_\mu b_{\nu-\mu} \right\}.
\]

A comparison of (1) and (6) shows that the correspondence (2) is an isomorphism under convolution; we embed \(S\) in \(\mathcal{F}\), identify \(S\) with \(\mathcal{F}_0\), elevate (2) to an
equality, and permit any notation for a member of \( S \) to be used as a notation for the corresponding member of \( \mathfrak{S}_0 \). In particular

\[
1 = \{ \cdots, 0, 0, 0 | 1, 0, 0, 0, \cdots, 0, \cdots \}
\]

\[
\sigma = \{ \cdots, 0, 0, 0 | 1, 1, 1, \cdots, 1, \cdots \}
\]

\[
\tau = \{ \cdots, 0, 0, 0 | 0, 1, 0, 0, 0, \cdots, 0, 0, \cdots \}
\]

\[
\tau^m = \{ \cdots, 0, 0, 0 | 0, 0, \cdots, 0, 1, 0, 0, 0, \cdots \}
\]

where \( m \) is a positive integer.

Defining \( \xi \) by:

\[
(7) \quad \xi = \{ 1 | 0, 0, 0, \cdots, 0, \cdots \}
\]

we have

\[
(8) \quad \xi^m = \{ 1, 0, 0, 0, \cdots, 0 | 0, 0, 0, \cdots, 0, \cdots \}
\]

where \( m \) is a positive integer.

Equations (7) and (9) may be verified by induction. Using (5) we may verify that

\[
\tau^m = 1
\]

\[
\tau^{m+n} = \xi^m \tau^n
\]

\[
\xi^{m+n} = \xi^m \xi^n
\]

\[
\tau^m[a_r] = \{a_{r-m}\}
\]

\[
\xi^m[a_r] = \{a_{r+m}\}
\]

Under ordinary addition and convolution multiplication \( \mathfrak{S} \) is a field. We need only verify here that each nonzero member of \( \mathfrak{S} \) has a multiplicative inverse. To begin with, every sequence of the form

\[
\{ a_0, a_1, a_2, \cdots \}
\]

in which \( a_0 \neq 0 \) (the sequence enters at the origin) has an inverse:

\[
\{ x_0, x_1, x_2, \cdots \}
\]

which may be evaluated as follows:

\[
\{ a_0, a_1, a_2, \cdots \} \{ x_0, x_1, x_2, \cdots \} = \{ 1, 0, 0, 0, \cdots \}
\]

\[
a_0x_0 = 1
\]

\[
a_0x_1 + a_1x_0 = 0
\]

\[
a_0x_2 + a_1x_1 + a_2x_0 = 0
\]

\[\vdots\]

Since the only division involved in solving for the \( x \)'s is division by \( a_0 \), and \( a_0 \neq 0 \), the \( x \)'s exist and so the desired inverse exists.

Finally, let \( a \) be any nonzero member of \( \mathfrak{S} \) which does not enter at the origin.
Since \( a \) is an entering sequence, there exists a sequence \( A \) and a positive integer \( m \) such that either

\[
a = \tau^mA \quad \text{or} \quad a = \xi^mA
\]

where \( A \) enters at the origin, and so has an inverse \( A^{-1} \) by the preceding paragraph. Then either

\[
(A^{-1}\xi^m)a = 1 \quad \text{or} \quad (A^{-1}\tau^m)a = 1
\]

and so, in any case, \( a \) has a multiplicative inverse, and \( \mathcal{F} \) is a field.

Since \( \mathcal{F} \) contains no divisors of zero, products lead to the introduction of fractions:

\[
\begin{array}{c}
a, b, c \in \mathcal{F} \\
\text{and} \\
ab = c \\
\text{and} \\
a \neq 0
\end{array}
\implies
\begin{array}{c}
\frac{c}{a} \text{ exists as a member of } \mathcal{F} \\
\text{and} \\
\frac{c}{a} = b \\
\text{and} \\
a \left( \frac{c}{a} \right) = c.
\end{array}
\]

In particular

\[
\zeta = \{1, 0, 0, 0, \cdots\} = \frac{1}{\tau} = \frac{1, 0, 0, 0, \cdots}{0, 1, 0, 0, 0, \cdots}
\]

and \( 1/\tau \) exists as a member of \( \mathcal{F} \).

Members of \( \mathcal{F} \) may be put into operational form in terms of \( \sigma, \tau, \) and/or \( \zeta \).

**Example 1.**

\[
\left\{ \frac{\nu(\nu - 1)}{2} u(\nu + 2) \right\} = \{3, 1 | 0, 0, 1, 3, 6, 10, 15, \cdots \}
\]

\[
= \zeta^2 \left\{ \frac{(\nu - 2)(\nu - 3)}{2} u(\nu) \right\} = \frac{\zeta^2}{2} \{ (\nu^2 - 5\nu + 6)u(\nu) \}
\]

\[
= \frac{\zeta^2}{2} (\sigma^2\tau + 2\sigma^3\tau^2 - 5\sigma^2\tau + 6\sigma)
\]

where \( \{\nu u(\nu)\} = \sigma^2\tau \) and \( \{\nu^2 u(\nu)\} = \sigma^2\tau + 2\sigma^3\tau^2 \) as shown in [2], and as may be checked straightforwardly. Then

\[
\left\{ \frac{\nu(\nu - 1)}{2} u(\nu + 2) \right\} = \sigma^3 - 2\sigma^2\zeta + 3\sigma\zeta^2.
\]

In Traub [6, p. 196], every quotient of “generalized” sequences with a nonzero denominator equals a shift operator times an ordinary sequence. Thus, in Traub’s notation,

\[
\frac{f}{g} = \frac{\hat{f}}{\omega e} = \omega^{-4} \frac{f}{e}
\]
where \( f/e \) equals an ordinary sequence since \( e \) assigns a nonzero value to the origin; \( \omega^{-i} \) is a shift operator, and is a "generalized" sequence—an ordered pair of ordinary sequences. In comparison, in the present paper, we are dealing with entering sequences (defined on \( J \)) instead of ordered pairs, and every quotient, \( b/a \), of entering sequences (with nonzero denominator) equals an entering sequence. In evaluating \( b/a \) we may replace \( a \), as in (10), by \( \tau^n A \) or \( \xi^n A \), as appropriate, and obtain respectively

\[
\frac{b}{a} = \xi^n \frac{b}{A} \quad \text{or} \quad \frac{b}{a} = \tau^n \frac{b}{A}
\]

where \( b/A, \xi^n, \) and \( \tau^n \) are all entering sequences.

**Example 2.**

\[
\frac{\{1, -1, 1, -1, 1, -1, \cdots\}}{\{1, 1, 1 | 1, 1, \cdots\}} = \frac{1}{1 + \tau} = \frac{1}{1 + \tau} \frac{1 - \tau}{\tau^3} = \frac{\tau^3}{1 + \tau} \{1, -2, 2, -2, 2, -2, \cdots\} = \{0, 0, 0, 1, -2, 2, -2, 2, -2, \cdots\}.
\]

**Example 3.**

\[
\frac{\{3, 1 | 0, 0, 1, 3, 6, 10, 15, \cdots\}}{\{0, 0, 0, 1, 3, 3, 1, 0, 0, 0, \cdots\}} = \frac{\sigma^3 - 2\sigma^2 \xi + 3\sigma \xi^2}{\tau^3 (\tau + 1)^3} \quad (\text{see example 1})
\]

\[
= \frac{1}{(1 - \tau)^3} - 2 \frac{1}{(1 - \tau)^2} \frac{1}{\tau} + 3 \frac{1}{1 - \tau} \frac{1}{\tau^3} = \frac{(6 \tau^3 - 8 \tau^4 + 3 \tau^5)}{(1 - \tau)^3} \quad (\text{omitting several algebraic steps})
\]

\[
= (6\tau^3 - 8\tau^4 + 3\tau^5) \{1, 0, 3, 0, 6, 0, 10, 0, 15, 0, \cdots\} \quad (\text{which may be checked by cross multiplication})
\]

\[
= \{6, 0, 18 | 0, 36, 0, 60, \cdots\}
\]

\[
+ \{ -8, 0, -24, 0 | -48, 0, -80, 0, \cdots\}
\]

\[
+ \{ 3, 0, 9, 0, 18 | 0, 30, 0, 45, \cdots\}
\]

\[
= \{3, -8, 15, -24, 36 | -48, 66, -80, 105, \cdots\}.
\]

The last result may be checked by cross multiplication:

\[
\{0, 0, 0, 1, 3, 3, 1, 0, 0, 0, \cdots\} \{3, -8, 15, -24, 36 | -48, 66, -80, 105, \cdots\} = \{3, 1 | 0, 0, 1, 3, 6, 10, 15, \cdots\}.
\]

A convenient way to multiply two entering sequences is to ignore the vertical lines at first, and then insert a vertical line in the final answer, following rules similar to those for the insertion of a decimal point in a product of decimals.
George Boole’s operator, $E$, [4, p. 16] which shifts a sequence to the left and replaces by zero the terms which pass the origin, operates only on sequences which vanish to the left of the origin:

$$E^n[f(\nu)u(\nu)] = [f(\nu + n)u(\nu)], \quad n = \text{nonnegative integer}.$$  

Thus $E$ cannot be identified with $\xi$; neither is $E$ to be discarded, since there is no convolution product to do the job that $E$ does, and that job is important. However, George Boole’s symbolic method [4, p. 215] is salvaged if $E$ is replaced by $\xi$ as discussed in [2]. Thus, Boole’s symbolic equation [4, pp. 217, 218]

$$bx \frac{E}{E - a} bx$$

becomes:

$$(11) \quad \frac{b'u(\nu)}{\xi - a} = \frac{b'u(\nu)}{b - a} + \frac{a'u(\nu)}{a - b}.$$  

This follows from the equation

$$\{c'u(\nu)\} = \frac{1}{1 - cr} = \frac{c'}{c}, \quad c = \text{number}$$

which is easily checked by cross multiplication. To prove (11) we have

$$\frac{b'u(\nu)}{\xi - a} = \frac{\xi}{\xi - b} \frac{1}{\xi - a} = \frac{1}{b - a} \frac{\xi}{\xi - b} + \frac{1}{a - b} \frac{\xi}{\xi - a}$$

$$= \frac{b'u(\nu)}{b - a} + \frac{a'u(\nu)}{a - b}.$$  

When operational forms of sequences are expressed in terms of $\xi$ they match the $Z$-transforms of sequences as used, for example, by Aseltine [5] (hence the use of "$\xi" for the reciprocal of $\tau$). For example [5, p. 259]

$$\{u(\nu)\} = \sigma = \frac{1}{1 - \tau} = \frac{\xi}{\xi - 1}.$$  

But now $\xi$ is a sequence and not a variable, a formula in $\xi$ equals a sequence rather than being a "transform" of it, and the introduction of the $\xi$-forms requires no theory of convergence of power series. In [2, pp. 140–143] it is shown that results previously obtained using the theory of functions of a complex variable including branch cuts and the theory of residues, may be obtained by purely algebraic methods from the field properties of $\xi$.

California State Polytechnic College
Pomona, California


