A Division Algebra for Sequences Defined on all the Integers

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The convolution ring, $S$, of sequences defined on the nonnegative integers, and the embedding of this ring in a field, have been discussed by Brand [1], Moore [2], [3], Traub [6], and others. Brand [1] specifically mentions that the field in which he embeds $S$ is a field of ordered pairs of members of $S$. Traub does not identify his field and does not mention "ordered pairs", but he mentions an analogy to Mikusiński's work [7], and so he probably had in mind the same field of ordered pairs as did Brand. In [2] this writer showed that it was not necessary to create such a field of ordered pairs since there already existed a more natural, less abstract field in which to embed $S$. It is the purpose of this article to introduce this already existing and more natural field, $F$, in which $S$ may be embedded.

It will be assumed that the reader is familiar with the convolution algebra of sequences as given in [1], [3], and [6] to the point of recognizing $S$ as an integral domain in which convolution products defined by

$$
(1) \quad \{a_r\} \{b_r\} = \left\{ \sum_{\mu=0}^{r} a_{\mu} b_{r-\mu} \right\}
$$

contain no divisors of zero, in which the multiplicative unity is the sequence

$$
\{1, 0, 0, 0, \ldots, 0, \ldots\},
$$

in which sequences of the form

$$
\{c, 0, 0, 0, \ldots, 0, \ldots\}
$$

behave like numbers and are identified with numbers:

$$
c = \{c, 0, 0, 0, \ldots, 0, \ldots\},
$$

in which the sequence

$$
\{0, 1, 0, 0, 0, \ldots, 0, \ldots\}
$$

is a shift operator denoted by "$\tau"", in which the sequence

$$
\{1, 1, 1, \ldots, 1, \ldots\}
$$

is a summing operator denoted by "$\sigma"", and in which members of $S$ have operational forms in terms of $\tau$ and/or $\sigma$.

The sequences $\sigma$ and $\tau$ are related by the equation

$$
\sigma (1 - \tau) = 1
$$

and since $S$ has no divisors of zero we introduce fractions and write (for example)

$$
\frac{1}{\sigma} = 1 - \tau = \{1, -1, 0, 0, 0, \ldots\}.
$$


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The fraction $1/\tau$, for example, does not exist as a member of $S$. But $1/\tau$ will exist as a member of $\mathcal{S}$.

Let $\mathcal{S}$ be the class of number valued sequences defined over the integral domain, $J$, each of which assigns at most a finite number of nonzero values to negative integers. For each member of $\mathcal{S}$ there is a least integer, $m$, to which the sequence assigns a nonzero value; the sequence will be said to enter at $m$, and the members of $\mathcal{S}$ will be called entering sequences. Equality, sums, and products with numbers, of members of $\mathcal{S}$ are defined in the usual termwise way. A suggested notation for such a sequence is

$$\{1, 2, 3 | 4, 5, 6, \ldots\}$$

where the vertical line—playing a role like a decimal point—separates values assigned to the negative integers from values assigned to the nonnegative integers, and the zeros assigned on the left are omitted for convenience.

Let $\nu$ be a variable on $J$. We define the unit step formula "$u(\nu)$":

$$u(\nu) = \begin{cases} 0, & \nu < 0, \\ 1, & \nu \geq 0. \end{cases}$$

Then $\{u(\nu)\}$ and $\{(\nu + 1)u(\nu)\}$ (for example) are members of $\mathcal{S}$, whereas $\{\nu + 1\}$ is not. The braces serve to bind out "$\nu$" converting a formula into a notation for a sequence.

There is a natural one-to-one correspondence between $S$ (sequences defined on the nonnegative integers) and the subclass, $\mathcal{S}_0$, of $\mathcal{S}$ consisting of sequences which enter at nonnegative points:

$$\{a_0, a_1, a_2, \ldots\} \leftrightarrow \{\ldots, 0, 0 | a_0, a_1, a_2, \ldots\}.$$  \hfill (2)

The convolution, or convolution product, of two sequences $a$ and $b$ from $\mathcal{S}$ is defined by

$$ab = \left\{ \sum_{\mu=-\infty}^{+\infty} a_\mu b_{\nu-\mu} \right\}.$$ \hfill (3)

If $c$ enters at $\alpha$ or to the right of $\alpha$ and $b$ enters at $\beta$ or to the right of $\beta$, then

$$(ab)_\nu = \begin{cases} \sum_{\mu=a}^{\nu} a_\mu b_{\nu-\mu}, & \nu \geq \alpha + \beta, \\ 0, & \nu < \alpha + \beta, \end{cases}$$ \hfill (4)

$$ab = \left\{ u(\nu - \alpha - \beta) \sum_{\mu=\alpha}^{\nu} a_\mu b_{\nu-\mu} \right\}.$$ \hfill (5)

The summation limits are finite in (5) since the sequences are entering sequences. In particular, if $a$ and $b$ are members of the subclass $\mathcal{S}_0$, we may take $\alpha = \beta = 0$ in (5) and (5) becomes

$$ab = \left\{ u(\nu) \sum_{\mu=0}^{\nu} a_\mu b_{\nu-\mu} \right\}.$$ \hfill (6)

A comparison of (1) and (6) shows that the correspondence (2) is an isomorphism under convolution; we embed $S$ in $\mathcal{S}$, identify $S$ with $\mathcal{S}_0$, elevate (2) to an
equality, and permit any notation for a member of $S$ to be used as a notation for the corresponding member of $\mathcal{S}_0$. In particular

$$
1 = \{ \cdots, 0, 0, 0 | 1, 0, 0, 0, \cdots, 0, \cdots \}
$$

$$
\sigma = \{ \cdots, 0, 0, 0 | 1, 1, 1, \cdots, 1, \cdots \}
$$

$$
\tau = \{ \cdots, 0, 0, 0 | 0, 1, 0, 0, 0, \cdots, 0, \cdots \}
$$

$$
\tau^m = \{ \cdots, 0, 0, 0 | 0, 0, \cdots, 0, 1, 0, 0, 0, \cdots \} \quad m \text{ zeros}
$$

Defining $\xi$ by:

$$
\xi = \{ 1 | 0, 0, 0, \cdots, 0, \cdots \}
$$

we have

$$
\xi^m = \left\{ 1, 0, 0, 0, \cdots, 0 \right\} \left\{ 0, 0, 0, \cdots, 0, \cdots \right\} \quad m \text{ positive integer.}
$$

Equations (7) and (9) may be verified by induction. Using (5) we may verify that

$$
\tau^n = 1
$$

$$
\tau^{m+n} = \xi^m \tau^n
$$

$$
\xi^{m+n} = \xi^m \xi^n \quad m, n \text{ positive integers.}
$$

$$
\tau^m [a_r] = \{ a_{r-m} \}
$$

$$
\xi^m [a_r] = \{ a_{r+m} \}
$$

Under ordinary addition and convolution multiplication $\mathcal{S}$ is a field. We need only verify here that each nonzero member of $\mathcal{S}$ has a multiplicative inverse. To begin with, every sequence of the form

$$
\{ a_0, a_1, a_2, \cdots \}
$$

in which $a_0 \neq 0$ (the sequence enters at the origin) has an inverse:

$$
\{ x_0, x_1, x_2, \cdots \}
$$

which may be evaluated as follows:

$$
\{ a_0, a_1, a_2, \cdots \} \{ x_0, x_1, x_2, \cdots \} = \{ 1, 0, 0, 0, \cdots \}
$$

$$
a_0 x_0 = 1
$$

$$
a_0 x_1 + a_1 x_0 = 0
$$

$$
a_0 x_2 + a_1 x_1 + a_2 x_0 = 0
$$

$$
\vdots
$$

Since the only division involved in solving for the $x$'s is division by $a_0$, and $a_0 \neq 0$, the $x$'s exist and so the desired inverse exists.

Finally, let $a$ be any nonzero member of $\mathcal{S}$ which does not enter at the origin.
Since \( a \) is an entering sequence, there exists a sequence \( A \) and a positive integer \( m \) such that either
\[
(10) \quad a = \tau^m A \quad \text{or} \quad a = \zeta^m A
\]
where \( A \) enters at the origin, and so has an inverse \( A^{-1} \) by the preceding paragraph. Then either
\[
(A^{-1}\tau^m) a = 1 \quad \text{or} \quad (A^{-1}\zeta^m) a = 1
\]
and so, in any case, \( a \) has a multiplicative inverse, and \( \mathcal{F} \) is a field.

Since \( \mathcal{F} \) contains no divisors of zero, products lead to the introduction of fractions:
\[
\begin{align*}
a, b, c & \in \mathcal{F} \\
\text{and} & \\
ab & = c \\
\Rightarrow & \\
\frac{c}{a} & = b
\end{align*}
\]
and
\[
\begin{align*}
a & \neq 0 \\
\Rightarrow & \\
a \left( \frac{c}{a} \right) & = c.
\end{align*}
\]
In particular
\[
\zeta = \{1 \mid 0, 0, 0, \cdots \} = \frac{1}{\tau} = \frac{\{1, 0, 0, 0, \cdots \}}{\{0, 1, 0, 0, 0, \cdots \}}
\]
and \( 1/\tau \) exists as a member of \( \mathcal{F} \).

Members of \( \mathcal{F} \) may be put into operational form in terms of \( \sigma \), \( \tau \), and/or \( \zeta \).

**Example 1.**
\[
\left\{ \frac{\nu(\nu - 1)}{2} u(\nu + 2) \right\} = \{3, 1 \mid 0, 0, 1, 3, 6, 10, 15, \cdots \}
\]
\[
= \zeta^2 \left\{ \frac{(\nu - 2) (\nu - 3)}{2} u(\nu) \right\} = \frac{\zeta^2}{2} \{ (\nu^2 - 5\nu + 6) u(\nu) \}
\]
\[
= \frac{\zeta^2}{2} \left( \sigma^2 \tau + 2\sigma^3 \tau^2 - 5\sigma^2 \tau + 6\sigma \right)
\]
where \( \{\nu u(\nu)\} = \sigma^2 \tau \) and \( \{\nu^2 u(\nu)\} = \sigma^2 \tau + 2\sigma^3 \tau^2 \) as shown in [2], and as may be checked straightforwardly. Then
\[
\left\{ \frac{\nu(\nu - 1)}{2} u(\nu + 2) \right\} = \sigma^3 - 2\sigma^2 \zeta + 3\sigma \zeta^2.
\]

In Traub [6, p. 196], every quotient of "generalized" sequences with a nonzero denominator equals a shift operator times an ordinary sequence. Thus, in Traub's notation,
\[
\frac{f}{g} = \frac{f}{\omega e} = \omega^{-1} \frac{f}{e}
\]
where \( f/e \) equals an ordinary sequence since \( e \) assigns a nonzero value to the origin; \( \omega^{-1} \) is a shift operator, and is a "generalized" sequence—an ordered pair of ordinary sequences. In comparison, in the present paper, we are dealing with entering sequences (defined on \( J \)) instead of ordered pairs, and every quotient, \( b/a \), of entering sequences (with nonzero denominator) equals an entering sequence. In evaluating \( b/a \) we may replace \( a \), as in (10), by \( \tau^n A \) or \( \zeta^n A \), as appropriate, and obtain respectively

\[
\frac{b}{a} = \tau^m b \quad \text{or} \quad \frac{b}{a} = \zeta^m b \quad \text{or} \quad \frac{b}{a} = \tau^m A
\]

where \( b/A, \zeta^m \), and \( \tau^m \) are all entering sequences.

Example 2.

\[
\frac{\{1, 1, 1, -1, 1, -1, \cdots\}}{\{1, 1, 1 | 1, 1, \cdots\}} = \frac{1/(1 + \tau)}{\tau^2 (1 + \tau)} = \frac{\tau^3}{1 - \tau} = \tau^3 \{1, 2, 2, 2, 2, 2, \cdots\} = \{0, 0, 0, 1, -2, 2, -2, 2, 2, \cdots\}.
\]

Example 3.

\[
\frac{\{3, 1 | 0, 0, 1, 3, 6, 10, 15, \cdots\}}{\{0, 0, 0, 1, 3, 3, 1, 0, 0, \cdots\}} = \frac{\sigma^3 - 2\sigma^2 \zeta + 3\sigma \zeta^2}{\tau^3 (1 + \tau)^3} \quad \text{(see example 1)}
\]

\[
= \frac{1}{1 - \tau^3} - 2 \frac{1}{1 - \tau^2} \frac{1}{1 - \tau^3} + 3 \frac{1}{1 - \tau^3} \frac{1}{1 - \tau^3}
\]

\[
= \frac{6}{1 - \tau^3} - 8 \frac{1}{1 - \tau^3} = (6\tau^3 - 8\tau + 3\tau^3) \{1, 0, 3, 0, 6, 0, 10, 0, 15, 0, \cdots\}
\]

\[
(\text{which may be checked by cross multiplication})
\]

\[
= \{6, 0, 18 | 0, 36, 0, 60, \cdots\}
\]

\[
+ \{ -8, 0, -24, 0 | -48, 0, -80, 0, \cdots\}
\]

\[
+ \{3, 0, 9, 0, 18 | 0, 30, 0, 45, \cdots\}
\]

\[
= \{3, -8, 15, -24, 36 | -48, 66, -80, 105, \cdots\}.
\]

The last result may be checked by cross multiplication:

\[
\{0, 0, 0, 1, 3, 3, 1, 0, 0, 0, \cdots\} \{3, -8, 15, -24, 36 | -48, 66, -80, 105, \cdots\}
\]

\[
= \{3, 1 | 0, 0, 1, 3, 6, 10, 15, \cdots\}.
\]

A convenient way to multiply two entering sequences is to ignore the vertical lines at first, and then insert a vertical line in the final answer, following rules similar to those for the insertion of a decimal point in a product of decimals.
George Boole's operator, $E$, [4, p. 16] which shifts a sequence to the left and replaces by zero the terms which pass the origin, operates only on sequences which vanish to the left of the origin:

$$E^n[f(\nu)u(\nu)] = [f(\nu + n)u(\nu)], \quad n = \text{nonnegative integer}. $$

Thus $E$ cannot be identified with $\xi$; neither is $E$ to be discarded, since there is no convolution product to do the job that $E$ does, and that job is important. However, George Boole's symbolic method [4, p. 215] is salvaged if $E$ is replaced by $\xi$ as discussed in [2]. Thus, Boole's symbolic equation [4, pp. 217, 218]

$$b^x

\frac{E - a}{b - a} = \frac{ca^x}{a - b}, \quad c = \text{arbitrary constant } a, b \text{ numbers}

$$

becomes:

$$\frac{[b'u(\nu)]}{\xi - a} = \frac{[b'u(\nu)]}{b - a} + \frac{[a'u(\nu)]}{a - b}. \quad (11)

$$

This follows from the equation

$$\{c'u(\nu)\} = \frac{1}{1 - cr} = \frac{\xi}{\xi - c}, \quad c = \text{number}

$$

which is easily checked by cross multiplication. To prove (11) we have

$$\frac{[b'u(\nu)]}{\xi - a} = \frac{\xi}{\xi - b} \frac{1}{\xi - a} = \frac{1}{b - a} \frac{\xi}{\xi - b} + \frac{1}{a - b} \frac{\xi}{\xi - a}

\frac{[b'u(\nu)]}{b - a} + \frac{[a'u(\nu)]}{a - b}.

$$

When operational forms of sequences are expressed in terms of $\xi$ they match the $Z$-transforms of sequences as used, for example, by Aseltine [5] (hence the use of "$\xi" for the reciprocal of $\tau$). For example [5, p. 259]

$$[u(\nu)] = \sigma = \frac{1}{1 - \tau} = \frac{\xi}{\xi - 1}.

$$

But now $\xi$ is a sequence and not a variable, a formula in $\xi$ equals a sequence rather than being a "transform" of it, and the introduction of the $\xi$-forms requires no theory of convergence of power series. In [2, pp. 140-143] it is shown that results previously obtained using the theory of functions of a complex variable including branch cuts and the theory of residues, may be obtained by purely algebraic methods from the field properties of $\xi$.

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