number of primes in the above expression for $h_N$. It should be borne in mind that the approximant vanishes at the endpoints of the interval $[0, \pi]$; consequently if the approximant does not have this property, we should modify it accordingly; this may involve subtracting a linear trend as suggested in similar circumstances by Lanczos [3, p. 236].

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A Note on Best Approximation in $E^n$

By J. T. Day

Let $D$ be a closed convex set with positive volume $V$ in Euclidean $n$-dimensional space. Let $f$ be a nonnegative function of class $C^2$ on $D$ (see [2]), and $Q$ be a linear polynomial on $D$, i.e.

$$Q(x) = a_0 + a_1x_1 + a_2x_2 + \cdots + a_nx_n, \quad x \in D.$$  

We consider the problem of "best" one sided approximation of $f$ by $Q$ in the sense that among all linear functions $Q(x)$ satisfying

$$Q(x) \leq f(x), \quad x \in D,$$

we are looking for that one which maximizes $\int_D Q \, dx$.

**Theorem 1.** The problem under consideration has a unique solution given by the tangent plane through the centroid $p$ of $D$, provided that the eigenvalues of the Hessian matrix $(f_{ij}(x))$, $x \in D$, are nonnegative.

The proof is by construction. Let the centroid $p$ of $D$ have cartesian coordinates $(p_1, p_2, \cdots, p_n)$. Then

$$\int_D Q \, dx = V \cdot Q(p_1, p_2, \cdots, p_n)$$

for all linear polynomials $Q$ (see [3]). Since $Q(p) \leq f(p)$, we choose $Q^*(p) = f(p)$. Choose $Q_1^*(p) = f_1(p)$, $Q_2^*(p) = f_2(p)$, \cdots, $Q_n^*(p) = f_n(p)$. Here $f_1(x) = (\partial f/\partial x_1)(x)$, etc. The above conditions determine $Q^*(x)$.

By Taylor's theorem we have $f(x) = Q^*(x) + R(x, p)$. The remainder $R(x, p)$ is nonnegative, since the eigenvalues of the Hessian matrix are nonnegative (see [2]). Thus $f(x) \geq Q^*(x)$. We conclude that $Q^*(x)$ is a "best" approximate.

Suppose there were another "best" approximate $T(x)$. Then $T(p)$ must equal $f(p)$. Consider a point $x = (x_1, p_2, \cdots, p_n)$ where $x_1 > p_1$. By Taylor's theorem we have
(3) \( f(x) = f(p) + f_1(p)(x_1 - p_1) + f_{11}(p_1 + s h, p_2, \cdots, p_n)(x_1 - p_1)^2/2. \)

Here \( h = x_1 - p_1, 0 < s < 1. \)

(4) \( T(x) = f(p) + T_1(p)(x_1 - p_1). \)

Since \( f(x) \geq T(x), \) we find that

(5) \( f_1(p) - T_1(p) + f_{11}(p_1 + s h, p_2, \cdots, p_n)(x_1 - p_1)/2 \geq 0. \)

The quantity \( f_1(p) - T_1(p) \) must be nonnegative, for otherwise we could choose \( (x_1 - p_1) \) so small that (5) could not hold. (We note here \( f_{11}(x) \geq 0 \) for \( x \in D \) by hypothesis.) A similar consideration in the case where \( p_1 > x_1 \) shows that \( f_1(p) = T_1(p) \). In the same manner one can show that

(6) \( f_i(p) = T_i(p), i = 2, \cdots, n. \)

Thus \( Q^*(x) \) and \( T(x) \) are identical.

The idea for this note occurred to the author after hearing a lecture by Prof. Ranko Bojanic [1] on "best" one sided approximation in the case of functions of one variable.

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1. R. Bojanic, "On polynomials of best one sided approximation." (To appear.)

A Close Approximation Related to the Error Function*

By Roger G. Hart

A function has been found that closely approximates the integral function

\[ F(x) = \int_x^\infty \exp\left(-t^2/2\right) dt \]

for all real values of \( x. \)

Let

\[ P(x) = \exp\left(-x^2/2\right) \left[ 1 - \frac{(1 + bx^2)^{1/2}/(1 + ax^2)}{P_0 x + [P_0 x^2 + \exp(-x^2/2)(1 + bx^2)^{1/2}/(1 + ax^2)]^{1/2}} \right] \]

\[ = P_0 + x^{-1}\left\{ \exp\left(-x^2/2\right) - \left[ P_0 x^2 + \exp\left(-x^2/2\right)(1 + bx^2)^{1/2}/(1 + ax^2)\right]^{1/2} \right\}, \]

where \( P_0 = (\pi/2)^{1/2} \approx 1.253314137, \)

\[ a = \frac{1 + (1 - 2\pi^2 + 6\pi)^{1/2}}{2\pi} \approx .212023887, \]

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