number of primes in the above expression for \( h_N \). It should be borne in mind that the approximant vanishes at the endpoints of the interval \([0, \pi]\); consequently if the approximant does not have this property, we should modify it accordingly; this may involve subtracting a linear trend as suggested in similar circumstances by Lanczos [3, p. 236].

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A Note on Best Approximation in \( E^n \)

By J. T. Day

Let \( D \) be a closed convex set with positive volume \( V \) in Euclidean \( n \)-dimensional space. Let \( f \) be a nonnegative function of class \( C^2 \) on \( D \) (see [2]), and \( Q \) be a linear polynomial on \( D \), i.e.

\[
Q(x) = a_0 + a_1 x_1 + a_2 x_2 + \cdots + a_n x_n, \quad x \in D.
\]

We consider the problem of "best" one sided approximation of \( f \) by \( Q \) in the sense that among all linear functions \( Q(x) \) satisfying

\[
(1) \quad Q(x) \leq f(x), \quad x \in D,
\]

we are looking for that one which maximizes \( \int_D Q \, dx \).

**Theorem 1.** The problem under consideration has a unique solution given by the tangent plane through the centroid \( p \) of \( D \), provided that the eigenvalues of the Hessian matrix \( f_{ij}(x) \), \( x \in D \), are nonnegative.

The proof is by construction. Let the centroid \( p \) of \( D \) have cartesian coordinates \((p_1, p_2, \cdots, p_n)\). Then

\[
(2) \quad \int_D Q \, dx = V \cdot Q(p_1, p_2, \cdots, p_n)
\]

for all linear polynomials \( Q \) (see [3]). Since \( Q(p) \leq f(p) \), we choose \( Q^*(p) = f(p) \). Choose \( Q_1^*(p) = f_1(p) \), \( Q_2^*(p) = f_2(p) \), \( \cdots \), \( Q_n^*(p) = f_n(p) \). Here \( f_1(x) = \frac{\partial f}{\partial x_1}(x) \), etc. The above conditions determine \( Q^*(x) \).

By Taylor's theorem we have \( f(x) = Q^*(x) + R(x, p) \). The remainder \( R(x, p) \) is nonnegative, since the eigenvalues of the Hessian matrix are nonnegative (see [2]). Thus \( f(x) \geq Q^*(x) \). We conclude that \( Q^*(x) \) is a "best" approximate.

Suppose there were another "best" approximate \( T(x) \). Then \( T(p) \) must equal \( f(p) \). Consider a point \( x = (x_1, p_2, \cdots, p_n) \) where \( x_1 > p_1 \). By Taylor's theorem we have

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(3) \( f(x) = f(p) + f_1(p)(x_1 - p_1) + f_{1i}(p_1 + sh, p_2, \ldots, p_n)(x_1 - p_1)^2/2. \)

Here \( h = x_1 - p_1, \ 0 < s < 1. \)

(4) \( T(x) = f(p) + T_1(p)(x_1 - p_1). \)

Since \( f(x) \geq T(x) \), we find that

(5) \( f_1(p) - T_1(p) + f_{1i}(p_1 + sh, p_2, \ldots, p_n)(x_1 - p_1)/2 \geq 0. \)

The quantity \( f_1(p) - T_1(p) \) must be nonnegative, for otherwise we could choose \( (x_1 - p_1) \) so small that (5) could not hold. (We note here \( f_{1i}(x) \geq 0 \) for \( x \in D \) by hypothesis.) A similar consideration in the case where \( p_1 > x_1 \) shows that \( f_i(p) = T_i(p), i = 2, \ldots, n. \) Thus \( Q^*(x) \) and \( T(x) \) are identical.

The idea for this note occurred to the author after hearing a lecture by Prof. Ranko Bojanic [1] on "best" one sided approximation in the case of functions of one variable.

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1. R. Bojanic, "On polynomials of best one sided approximation." (To appear.)

A Close Approximation Related to the Error Function*

By Roger G. Hart

A function has been found that closely approximates the integral function

\[ F(x) = \int_x^\infty \exp \left(-\frac{t^2}{2}\right) dt \]

for all real values of \( x. \)

Let

\[ P(x) = \frac{\exp \left(-\frac{x^2}{2}\right)}{x} \left[ 1 - \frac{(1 + bx^2)^{1/2}/(1 + ax^2)}{P_0 + \frac{x^2}{2} + \exp \left(-\frac{x^2}{2}\right)(1 + bx^2)^{1/2}/(1 + ax^2)^{1/2}} \right] \]

\[ = P_0 + x^{-1}\left[ \exp \left(-\frac{x^2}{2}\right) - \left\{ P_0 + \frac{x^2}{2} + \exp \left(-\frac{x^2}{2}\right)(1 + bx^2)^{1/2}/(1 + ax^2)^{1/2} \right\} \right], \]

where \( P_0 = (\pi/2)^{1/2} \approx 1.253314137, \)

\[ a = \frac{1 + (1 - 2\pi^2 + 6\pi)^{1/2}}{2\pi} \approx .212023887, \]

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