A Starting Method for Solving Nonlinear Volterra Integral Equations

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Abstract. In this paper a fifth order starting method is given for Volterra equations of the form \( y(t) = f(t) + \int_{x_0}^{t} k(t, s, y(s)) \, ds \). Computational examples are given for the method as a starting method for the Gregory-Newton method.

1. Introduction. In this paper we shall consider an \( O(h^5) \) starting method for the numerical solution of the nonlinear Volterra integral equation

\[
y(t) = f(t) + \int_{x_0}^{t} k(t, s, y(s)) \, ds.
\]

After stating our algorithm we shall discuss its derivation and consider some computational examples. In our computational examples we shall consider our method as a starting method for the Gregory-Newton method. The Gregory-Newton method in this context has been discussed by Fox and Goodwin [2], Noble [8], and Todd [11].

2. The Algorithm. The self-starting method described here advances the solution from \( x_0 \) to \( x_0 + h \), \( x_0 + h \) to \( x_0 + 2h \), \ldots, \( x_0 + 5h \) to \( x_0 + 6h \). To advance from \( x_0 \) to \( x_0 + h \) we compute

\[
\begin{align*}
\hat{y}_{1/3} &= f \left( x_0 + \frac{h}{3} \right) + \frac{h}{3} k \left( x_0 + \frac{h}{3}, x_0, y_0 \right), \\
y_{1/3} &= f \left( x_0 + \frac{h}{3} \right) + \frac{h}{6} \left[ k \left( x_0 + \frac{h}{3}, x_0, y_0 \right) + k \left( x_0 + \frac{h}{3}, x_0 + \frac{h}{3}, \hat{y}_{1/3} \right) \right], \\
\hat{y}_{2/3} &= f \left( x_0 + \frac{2h}{3} \right) + \frac{2h}{3} k \left( x_0 + \frac{2h}{3}, x_0 + \frac{h}{3}, y_{1/3} \right), \\
y_{2/3} &= f \left( x_0 + \frac{2h}{3} \right) + \frac{h}{3} \left[ k \left( x_0 + \frac{2h}{3}, x_0 + \frac{h}{3}, y_{1/3} \right) + k \left( x_0 + \frac{h}{3}, x_0 + \frac{h}{3}, \hat{y}_{2/3} \right) \right], \\
\hat{y}_1 &= f(x_0 + h) + \frac{h}{4} \left[ k(x_0 + h, x_0, y_0) + 3k \left( x_0 + h, x_0 + \frac{2h}{3}, \hat{y}_{2/3} \right) \right], \\
y_1 &= f(x_0 + h) + \frac{h}{6} \left[ k(x_0 + h, x_0, y_0) + 4k \left( x_0 + h, x_0 + \frac{h}{2}, \hat{y}_{1/2} \right) + k(x_0 + h, x_0 + h, \hat{y}_1) \right].
\end{align*}
\]

To advance from \( x_0 + h \) to \( x_0 + 2h \) we compute

\[
\hat{y}_{3/2} = f \left( x_0 + \frac{3h}{2} \right) + \frac{3h}{4} \left[ k \left( x_0 + \frac{3h}{2}, x_0 + \frac{h}{2}, \hat{y}_{1/2} \right) + k \left( x_0 + \frac{3h}{2}, x_0 + h, y_1 \right) \right],
\]

\[ y_{3/2} = f\left(x_0 + \frac{3h}{2}\right) + \frac{3h}{16} \left[ k\left(x_0 + \frac{3h}{2}, x_0, y_0\right) + 3k\left(x_0 + \frac{3h}{2}, x_0 + \frac{h}{2}, y_{1/2}\right) \right. \\
\left. + 3k\left(x_0 + \frac{3h}{2}, x_0 + h, y_1\right) + k\left(x_0 + \frac{3h}{2}, x_0 + \frac{3h}{2}, y_{3/2}\right) \right], \tag{9} \]

\[ \dot{y}_2 = f(x_0 + 2h) + \frac{2h}{3} \left[ k\left(x_0 + 2h, x_0 + \frac{h}{2}, y_{1/2}\right) \cdot 2 \\
- k(x_0 + 2h, x_0 + h, y_1) + 2k\left(x_0 + 2h, x_0 + \frac{3h}{2}, y_{3/2}\right) \right], \tag{10} \]

\[ y_2 = f(x_0 + 2h) + \frac{h}{6} \left[k(x_0 + 2h, x_0, y_0) + 4k(x_0 + h2, x_{1/2}, y_{1/2}) \right. \\
\left. + 2k(x_2, x_1, y_1) + 4k(x_2, x_{3/2}, y_{3/2}) + k(x_2, x_2, \dot{y}_2) \right]. \tag{11} \]

To advance from \( x_0 + 2h \) to \( x_0 + 3h \) we compute

\[ \dot{y}_{3/2} = f\left(x_0 + \frac{5h}{2}\right) + \frac{5h}{48} \left[ 11k\left(x_0 + \frac{5h}{2}, x_0 + \frac{h}{2}, y_{1/2}\right) \right. \\
\left. + k\left(x_0 + \frac{5h}{2}, x_0 + h, y_1\right) + k\left(x_0 + \frac{5h}{2}, x_0 + \frac{3h}{2}, y_{3/2}\right) \right. \\
\left. + 11k\left(x_0 + \frac{5h}{2}, x_0 + 2h, y_2\right) \right], \tag{12} \]

\[ y_{3/2} = f\left(x_0 + \frac{5h}{2}\right) + \frac{5h}{576} \left[ 19k\left(x_0 + \frac{5h}{2}, x_0, y_0\right) \right. \\
\left. + 75k\left(x_0 + \frac{5h}{2}, x_{1/2}, y_{1/2}\right) + 50k\left(x_0 + \frac{5h}{2}, x_1, y_1\right) \right. \\
\left. + 50k\left(x_0 + \frac{5h}{2}, x_{3/2}, y_{3/2}\right) + 75k(x_{5/2}, x_2, y_2) + 19k(x_{5/2}, x_{5/2}, \dot{y}_{5/2}) \right], \tag{13} \]

\[ \dot{y}_3 = f(x_0 + 3h) + \frac{3h}{20} \left[ 11k(x_3, x_{1/2}, \dot{y}_{1/2}) - 14k(x_3, x_1, y_1) \right. \\
\left. + 26k(x_3, x_{3/2}, y_{3/2}) - 14k(x_3, x_2, y_2) + 11k(x_3, x_{5/2}, y_{5/2}) \right], \tag{14} \]

\[ y_3 = f(x_0 + 3h) + \frac{h}{6} \left[k(x_3, x_0, y_0) + 4k(x_3, x_{1/2}, \dot{y}_{1/2}) + 2k(x_3, x_1, y_1) \right. \\
\left. + 4k(x_3, x_{3/2}, y_{3/2}) + 2k(x_3, x_2, y_2) + 4k(x_3, x_{5/2}, y_{5/2}) + k(x_3, x_3, \dot{y}_3) \right]. \tag{15} \]

To advance from \( x_0 + 3h \) to \( x_0 + 4h \) we compute

\[ \dot{y}_4 = f(x_0 + 4h) + \frac{4h}{3} \left[ 2k(x_4, x_1, y_1) - k(x_4, x_2, y_2) + 2k(x_4, x_3, y_3) \right], \tag{16} \]

\[ y_4 = f(x_0 + 4h) + \frac{4h}{90} \left[ 7k(x_0 + 4h, x_0, y_0) + 32k(x_0 + 4h, x_0 + h, y_1) \right. \\
\left. + 12k(x_0 + 4h, x_2, y_2) + 32k(x_4, x_3, y_3) + 7k(x_4, x_4, \dot{y}_4) \right]. \tag{17} \]
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To advance from \(x_0 + 4h\) to \(x_0 + 5h\) we compute

\[
\dot{y}_5 = f(x_0 + 5h) + \frac{5h}{24} \left[ 11k(x_5, x_1, y_1) + k(x_5, x_2, y_2) + k(x_5, x_3, y_3) + 11k(x_5, x_4, y_4) \right],
\]

\[
y_5 = f(x_0 + 5h) + \frac{5h}{288} \left[ 19k(x_0 + 5h, x_0, y_0) + 75k(x_0 + 5h, x_0 + h, y_1) + 50k(x_5, x_2, y_2) + 50k(x_5, x_3, y_3) + 75k(x_5, x_4, y_4) + 19k(x_5, x_5, y_5) \right].
\]

To advance from \(x_0 + 5h\) to \(x_0 + 6h\) we compute

\[
\dot{y}_6 = f(x_0 + 6h) + \frac{6h}{20} \left[ 11k(x_6, x_1, y_1) - 14k(x_6, x_2, y_2) + 26k(x_6, x_3, y_3) - 14k(x_6, x_4, y_4) + 11k(x_6, x_5, y_5) \right],
\]

\[
y_6 = f(x_0 + 6h) + \frac{3h}{10} \left[ k(x_6, x_0, y_0) + 5k(x_6, x_1, y_1) + k(x_6, x_2, y_2) + 6k(x_6, x_3, y_3) + k(x_6, x_4, y_4) + 5k(x_6, x_5, y_5) + k(x_6, x_6, y_6) \right].
\]

3. Derivation of Algorithm. We shall sketch the derivation of the algorithm. Many of the ideas for the algorithm will be found in a paper due to Kuntzmann [5].

If we approximate the integral in (1) by Simpson’s rule on the interval \([x_0, x_0 + h]\) we obtain

\[
y(x_0 + h) = f(x_0 + h) + \frac{h}{6} \left[ k(x_0 + h, x_0, y_0) + 4k(x_0 + h, x_0 + \frac{h}{2}, y(x_0 + \frac{h}{2})) + k(x_0 + h, x_0 + h, y(x_0 + h)) \right] - \frac{h^5}{2880} \left[ k(x_0 + h, \xi, y(\xi)) \right].
\]

where \(x_0 < \xi < x_0 + h\). Here \(y(x_0 + h/2)\) and \(y(x_0 + h)\) are not known in the right side of (22). If we are to use (22) we must obtain accurate approximate values for \(y(x_0 + h/2)\) and \(y(x_0 + h)\). We do this in the following manner. First we note that

\[
y(x_0 + h) = f(x_0 + h) + \frac{h}{4} \left[ k(x_0 + h, x_0, y_0) + 3k(x_0 + h, x_0 + \frac{2h}{3}, y(x_0 + \frac{2h}{3})) \right] + O(h^4)
\]

is an \(O(h^4)\) approximation to \(y(x_0 + h)\). (This is the Radau two-point rule.) However, here we do not know \(y(x_0 + 2h/3)\), but if we could obtain it to \(O(h^3)\) then we could use (23). Thus, we attempt to attain an \(O(h^3)\) approximation to \(y(x_0 + 2h/3)\). This is done by using the midpoint rule

\[
y \left( x_0 + \frac{2h}{3} \right) = f \left( x_0 + \frac{2h}{3} \right) + \frac{2h}{3} k \left( x_0 + \frac{2h}{3}, x_0 + \frac{h}{3}, y \left( x_0 + \frac{h}{3} \right) \right) + O(h^3).
\]
However here we do not know \( y(x_0 + h/3) \) to \( O(h^3) \). We obtain it to \( O(h^3) \) by using the trapezoidal rule and Taylor’s series

\[
y\left(x_0 + \frac{h}{3}\right) = f\left(x_0 + \frac{h}{3}\right) + \frac{h}{6} \left[ k\left(x_0 + \frac{h}{3}, x_0, y_0\right) + k\left(x_0 + \frac{h}{3}, x_0 + \frac{h}{3}, y_0\left(x_0 + \frac{h}{3}\right)\right)\right] + O(h^3),
\]

\[
y\left(x_0 + \frac{h}{3}\right) = f\left(x_0 + \frac{h}{3}\right) + \int_{x_0}^{x_0 + h/3} \left[ k\left(x_0 + \frac{h}{3}, x_0, y_0\right) + O(h)\right] ds
\]

\[
= f\left(x_0 + \frac{h}{3}\right) + \frac{h}{3} k\left(x_0 + \frac{h}{3}, x_0, y_0\right) + O(h^2).
\]

Summarizing the above procedure, we have that formula (23) is used to predict a value for \( y_1 \) (Eq. (6)) which is then corrected with (25) (Eq. (7)). Formula (26) is used to predict a value for \( y_{1/3} \) (Eq. (2)) which is corrected with (25) (Eq. (3)).

The value of \( y_{1/2} \) is obtained by approximating the integral in

\[
y\left(x_0 + \frac{h}{2}\right) = f\left(x_0 + \frac{h}{2}\right) + \int_{x_0}^{x_0 + h/2} k(t, s, y(s)) ds, \quad t = x_0 + \frac{h}{2},
\]

by the Radau two-point rule, disregarding the truncation error and substituting \( y_{1/3} \) in for \( y(x_0 + h/3) \).

In advancing from \( x_0 + h \) to \( x_0 + 2h \), we first let \( x \) equal to \( x_0 + 2h \) in (1) to obtain

\[
y(x_0 + 2h) = f(x_0 + 2h) + \int_{x_0}^{x_0 + 2h} k(x_0 + 2h, s, y(s)) ds.
\]

This integral could be evaluated by Simpson’s rule if we knew accurate approximate values for \( y_{3/2} \) and \( y_2 \). We obtain approximate values for \( y_{3/2} \) by first using the open Newton-Cotes formula

\[
y\left(x_0 + \frac{3h}{2}\right) = f\left(x_0 + \frac{3h}{2}\right) + \frac{3h}{4} \left[ k\left(x_0 + \frac{3h}{2}, x_0 + \frac{h}{2}, y_{1/2}\right) + k\left(x_0 + \frac{3h}{2}, x_0 + h, y_1\right)\right] + O(h^3)
\]

and substituting this value into Simpson’s three-eighths’ rule on \([x_0, x_0 + 3h/2]\)

\[
y\left(x_0 + \frac{3h}{2}\right) = f\left(x_0 + \frac{3h}{2}\right) + \frac{3h}{16} \left[ k(x_{3/2}, x_0, y_0) + 3k(x_{3/2}, x_{1/2}, y_{1/2})
\]

\[
+ 3k(x_{3/2}, x_1, y_1) + k(x_{3/2}, x_{3/2}, y_{3/2})\right] + O(h^4).
\]

An accurate value for \( y(x_0 + 2h) \) is obtained by using the Newton-Cotes open formula

\[
y(x_0 + 2h) = f(x_0 + 2h) + \frac{2h}{3} \left[ 2k(x_2, x_{1/2}, y_{1/2}) - k(x_0 + 2h, x_1, y_1)
\]

\[
+ 2k(x_2, x_{3/2}, y_{3/2})\right] + O(h^5).
\]
and substituting this result into Simpson's rule

\[ y(x_0 + 2h) = f(x_0 + 2h) + \frac{h}{6} \left[ k(x_2, x_0, y_0) + 4k(x_2, x_{1/2}, y_{1/2}) + 2k(x_2, x_1, y_1) \right. \]

\[ + 4k(x_2, x_{3/2}, y_{3/2}) + k(x_2, x_2, y_2) \left] + O(h^5). \right. \]

To advance from \( x_0 + 2h \) to \( x_0 + 3h \) we could again use Simpson's rule if we knew accurate approximate values for \( y_{5/2} \) and \( y_5 \). We proceed as follows. Use the open Newton-Cotes formula

\[ y_{5/2} = f\left(x_0 + \frac{5h}{2}\right) + \frac{5h}{48} \left[ 11k\left(x_0 + \frac{5h}{2}, x_{1/2}, y_{1/2}\right) + k\left(x_0 + \frac{5h}{2}, x_1, y_1\right) \right. \]

\[ + k\left(x_0 + \frac{5h}{2}, x_{3/2}, y_{3/2}\right) + 11k(x_{5/2}, x_2, y_2) \left] + O(h^5). \right. \]

along with the closed Newton-Cotes formula

\[ y_{5/2} = f\left(x_0 + \frac{5h}{2}\right) + \frac{5h}{576} \left[ 19k(x_{5/2}, x_0, y_0) + 75k(x_{5/2}, x_{1/2}, y_{1/2}) \right. \]

\[ + 50k(x_{5/2}, x_1, y_1) + 50k(x_{5/2}, x_{3/2}, y_{3/2}) \]

\[ + 75k(x_{5/2}, x_2, y_2) + 19k(x_{5/2}, x_{5/2}, \dot{y}_{5/2}) \left] + O(h^5). \right. \]

To obtain an approximate value for \( y \) at \( x_3 \) we use the open Newton-Cotes formula

\[ y_3 = f(x_0 + 3h) + \frac{3h}{20} \left[ 11k(x_0 + 3h, x_{1/2}, y_{1/2}) - 14k(x_3, x_1, y_1) \right. \]

\[ + 26k(x_3, x_{3/2}, y_{3/2}) - 14k(x_3, x_2, y_2) + 11k(x_3, x_{5/2}, y_{5/2}) \left] + O(h^5) \right. \]

together with Simpson's rule

\[ y(x_0 + 3h) = f(x_0 + 3h) + \frac{h}{6} \left[ k(x_0 + 3h, x_0, y_0) + 4k(x_3, x_{1/2}, y_{1/2}) \right. \]

\[ + 2k(x_3, x_1, y_1) + 4k(x_3, x_{3/2}, y_{3/2}) + 2k(x_3, x_2, y_2) \]

\[ + 4k(x_3, x_{5/2}, y_{5/2}) + k(x_3, x_3, y_3) \left] + O(h^5). \right. \]

It should be noted that the predictor is of higher order than the corrector here. To advance from \( x_0 + 3h \) to \( x_0 + 4h \) we approximate the integral in

\[ y(x_0 + 4h) = f(x_0 + 4h) + \int_{x_0}^{x_0+4h} k(x_0 + 4h, s, y(s)) \, ds \]

by the Newton-Cotes formula

\[ y(x_0 + 4h) = f(x_0 + 4h) + \frac{4h}{90} \left[ 7k(x_4, x_0, y_0) + 32k(x_4, x_1, y_1) + 12k(x_4, x_2, y_2) \right. \]

\[ + 32k(x_4, x_3, y_3) + 7k(x_4, x_4, y_4) \left] + O(h^7). \right. \]
Here \( y_4 \) is obtained from the open Newton-Cotes formula

\[
y(x_0 + 4h) = f(x_0 + 4h) + \frac{4h}{3} [2k(x_4, x_1, y_1) - k(x_4, x_2, y_2) + 2k(x_4, x_3, y_3)] + O(h^3).
\]

An approximate value of \( y \) at \( x_0 + 5h \) is obtained by the open Newton-Cotes formula

\[
y(x_0 + 5h) = f(x_0 + 5h) + \frac{5h}{24} [11k(x_5, x_1, y_1) + k(x_5, x_2, y_2)
+ k(x_5, x_3, y_3) + 11k(x_5, x_4, y_4)] + O(h^5)
\]

combined with the closed Newton-Cotes formulae

\[
y(x_0 + 5h) = f(x_0 + 5h) + \frac{5h}{288} [19k(x_5, x_0, y_0) + 75k(x_5, x_1, y_1)
+ 50k(x_5, x_2, y_2) + 50k(x_5, x_3, y_3)
+ 75k(x_5, x_4, y_4) + 19k(x_5, x_5, y_5)] + O(h^6).
\]

To advance from \( x_0 + 5h \) to \( x_0 + 6h \) we use the open Newton-Cotes formula

\[
y_6 = f(x_0 + 6h) + \frac{6h}{20} [11k(x_6, x_0, y_0) - 14k(x_6, x_1, y_1) + 26k(x_6, x_2, y_2)
+ 14k(x_6, x_3, y_3) + 11k(x_6, x_5, y_5)] + O(h^7)
\]

together with Weddle's rule

\[
y_6 = f(x_0 + 6h) + \frac{3h}{10} [k(x_6, x_0, y_0) + 5k(x_6, x_1, y_1) + k(x_6, x_2, y_2)
+ 6k(x_6, x_3, y_3) + k(x_6, x_4, y_4) + 5k(x_6, x_5, y_5) + k(x_6, x_6, y_6)] + O(h^7).
\]

The Newton-Cotes open and closed formulae and Weddle's rule are given in Milne [7]. For the other integration rules used here, see Hildebrand [3]. It should be noted that we have assumed that the eighth partial derivative of \( k \) with respect to \( s \) and \( y(s) \) exist and is bounded in order to apply our method.

The method under consideration can be applied to systems of integral equations.

4. Use of Gregory-Newton Formulae. The Gregory-Newton Formulae (see Todd [11], Hildebrand [3])

\[
\int_{x_0}^{x_0 + mh} f(p) \, dp = h \left\{ \frac{f(x_0)}{2} + f(x_1) + \cdots + f(x_{n-1}) + \frac{f(x_n)}{2} \right\}
+ \frac{h}{12} \left[ [f(x_1) - f(x_0)] - [f(x_n) - f(x_{n-1})] \right]
- \frac{h}{24} \left[ [f(x_1) - 2f(x_1) + f(x_0)] + [f(x_n) - 2f(x_{n-1}) + f(x_{n-2})] \right]
\]
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\[ + \frac{19h}{720} \{ [f(x_3) - 3f(x_2) + 3f(x_1) - f(x_0)] - [f(x_n) - 3f(x_{n-1})] \\
+ 3f(x_{n-2}) - f(x_{n-3}) ] \}

\[ - \frac{3h}{160} \{ [f(x_4) - 4f(x_3) + 6f(x_2) - 4f(x_1) + f(x_0)] \\
+ [f(x_n) - 4f(x_{n-1}) + 6f(x_{n-2}) - 4f(x_{n-3}) + f(x_{n-4})] \}

\[ + \frac{863h}{60480} [\Delta^5 f(x_0) - \nabla^5 f(x_n)] + \cdots \]

was used by Fox and Goodwin [2] in their treatment of linear Volterra integral equations. In this paper we use the Gregory-Newton formulae through fourth differences to advance the solution from \( x = x_0 + 6h \) to any \( x = x_0 + Nh \).

Since the integral equation is nonlinear, there is a need for a "predictor" to correspond to the role of the Gregory-Newton formula as "corrector." In our work we have used the following scheme. If we are to advance from \( x_0 + (2N - 1)h \) to \( x_0 + 2Nh \) use Simpson's rule with step size \( h \), from \( x_0 \) through \( x_0 + 2Nh - 4h \), then use the open Newton-Cotes formulae

\[ \int_{x_0}^{x_0+4h} \frac{3}{2} \left[ 2y_1 - y_2 + 2y_3 \right] + O(h^5) \]

on the interval \([x_0 + 2Nh - 4h, x_0 + 2Nh] \). In case \( x = x_0 + (2N - 1)h \) we first integrate from \( x_0 \) to \( x_0 + 3h \) with Simpson's "three-eighths" rule followed by Simpson's rule until we come to \( x_0 + (2N - 1)h - 4h \). Then apply the open Cotes formula used above. This predictor has enabled us to use the Gregory-Newton formula with only two iterations. Before using this, an \( O(h^2) \) predictor was used. However seven iterations were necessary in this case. Here the iterations were stopped after a certain number of decimal places of accuracy were achieved.

5. Computational Examples. The following computational examples were computed in Fortran (single precision) on the CDC 1604. By error we mean

\[ \text{error} = | \text{true} - \text{approximate value} |. \]

Example 1. The integral equation

\[ y(t) = 1 - t + \int_0^t (te^{t(2x^2)} + e^{-2x^2}) \cdot (y(x))^2 \, dx \]

has the solution \( y(x) = e^{x^2} \). It has been considered by Laudet and Oules [6]. We find the following errors.

Example 2. The integral equation

\[ y(t) = \frac{2t^{3/2}}{3} + \int_0^t (y(x))^{1/2} \, dx \]

was obtained by integrating the differential equation \( y' = x^{1/2} + y^{1/2}, \ y(0) = 0 \). This differential equation (see Todd [11], Noble [9]) does not possess a Taylor...
pansion about the origin. Its solution about the origin can be written in the series

\[
y(x) = \frac{2}{3} x^{3/2} + \frac{4}{7} \left(\frac{2}{3}\right)^{1/2} x^{7/4} + \frac{1}{7} x^2 + \frac{1}{49} \left(\frac{2}{3}\right)^{1/2} x^{9/4} - \frac{2}{1715} x^{5/2} + \cdots
\]

we obtain the following values for \( x \) at \(.1, .2, 1.0\) with step sizes \(.1, .05, .025\).

These values compare quite favorably with those obtained by Noble using the Runge-Kutta method (see Noble [9]).

**Example 3.** The integral equation

\[
y(t) = \int_0^t \max (x, y) \, dx
\]

was obtained from the differential equation \( y' = \max (x, y) \), \( y(0) = 0 \) (see Burkill [1]). The solution of this differential equation is

\[
y(x) = x^{3/2} \quad \text{for} \quad x \leq 2, \quad y(x) = 2e^{(x-2)} \quad \text{for} \quad x > 2.
\]

Thus there is a discontinuity in \( y'' \) at \( x = 2 \).

In this example somewhat better results in the region \( x \geq 2 \) were obtained by using the Runge-Kutta method.

**Example 4.** The integral equation

\[
y(t) = 2t + 3 + \int_0^t - y(x)(2(t - x) + 3) \, dx
\]

discussed by Todd [11]. The equation has the exact solution \( y(t) = 4e^{-2t} - e^{-t} \).

In addition to the above examples the writer has computed examples given by Jones [4], Pouzet [10], Fox and Goodwin [2] and others. These numerical examples are available from the writer in an MRC report.

### Table 1

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A STARTING METHOD FOR VOLTERRA EQUATIONS

Table 3

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