Confluent Expansions*

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I. Introduction and Summary. It is well known in special functions, see [1], that the confluent hypergeometric function is a limiting form of the Gaussian hypergeometric function, i.e.

\[
\lim_{b \to \infty} \, _2F_1\left( \frac{a}{c}, \frac{b}{c} \left| \frac{z}{b} \right. \right) = \, _1F_1\left( \frac{a}{c} \left| z \right. \right),
\]

or in series form,

\[
\lim_{b \to \infty} \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} \left( \frac{z}{b} \right)^k = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k k!} \left( \frac{z}{b} \right)^k, \quad \left( \sigma \right)_\mu = \frac{\Gamma(\sigma + \mu)}{\Gamma(\sigma)}.
\]

Note that if \( a = c \), (1.2) reduces to the familiar limit,

\[
\lim_{b \to \infty} \left( 1 - \frac{z}{b} \right)^{-b} = e^z.
\]

We will refer to the limit process in (1.2) as a confluence with respect to \( b \). More generally, we will refer to any limit process of the form \( \lim_{b \to \infty} \sum_{k=0}^{\infty} f_k(b) \), as a confluence with respect to \( b \), if the functions \( f_k(b) \), up to a multiplicative constant dependent on \( k \), are composed of a finite number of multiplicative factors of the form \((\pm b + \omega_1)(b + \omega_2)^k\) or their reciprocals, where \( \omega_1 \) and \( \omega_2 \) are constants independent of \( b \) and \( k \). The value of the limit, if it exists, will be called the confluent limit with respect to \( b \). The reference to \( b \) will occasionally be suppressed. As another example of a confluent limit in special functions, see [2], we quote the important classical relation between the Jacobi polynomials \( P_n^{(a,b)}(z) \), and the Bessel functions \( J_\alpha(z) \),

\[
\lim_{n \to \infty} n^{-\alpha} P_n^{(a,b)}\left( 1 - \frac{z^2}{2n^2} \right) = \left( \frac{z}{2} \right)^{-\alpha} J_\alpha(z),
\]

or in hypergeometric form,

\[
\lim_{n \to \infty} \frac{n^{-\alpha}(n + 1)\alpha}{\Gamma(1 + \alpha)} \, _2F_1\left( \frac{-n, n + \alpha + \beta + 1}{1 + \alpha} \left| \frac{z^2}{4n^2} \right. \right) = \frac{1}{\Gamma(1 + \alpha)} \, _0F_1\left( 1 + \alpha \left| -\frac{z^2}{4} \right. \right).
\]

In the situation where a confluence with respect to \( b \) is possible, it is of interest to consider what happens when \( b \) is large but finite. This leads in a natural way to expansions in inverse powers of \( b \) or a related variable. Such expansions may be
either analytic or asymptotic in nature, and will be referred to as analytic or asymptotic confluent expansions respectively, with respect to $b$. In this paper, several canonical types of confluent expansions will be examined.

For future reference, it is convenient to quote the following Tricomi and Erdélyi result [3].

**Theorem.** If $\alpha$ and $\beta$ are bounded quantities,

$$
\frac{\Gamma(z + \alpha)}{\Gamma(z + \beta)} \sim \sum_{j=0}^{\infty} \frac{(-1)^j (\beta - \alpha)_j}{j!} B_j^{(\alpha-\beta+1)}(\alpha) z^{\alpha-\beta-j},
$$

(1.6) 

$z \to \infty$, $|\arg(z + \alpha)| < \pi - \delta$, $\delta > 0$; $B_0^{(\alpha-\beta+1)}(\alpha) = 1,$

where the $B_j^{(\alpha-\beta+1)}(\alpha)$ are the generalized Bernoulli polynomials defined by

$$
\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{\alpha t} = \sum_{j=0}^{\infty} \frac{t^j}{j!} B_j^{(\alpha)}(x), \quad |t| < 2\pi.
$$

(1.7)

We remark that if $\beta - \alpha$ is an integer $\leq 0$, the asymptotic relation in (1.6) is exact, i.e., asymptotic equality ($\sim$) can be replaced by ordinary equality ($=$). Moreover, if $\beta - \alpha$ is an integer $> 0$, $|z| > \text{Max} \{|\alpha|, |\beta - 1|\}$, then the asymptotic relation in (1.6) is again exact.

II. Analytic Confluent Expansions. In this section we generalize the confluent limits in (1.1) and (1.5). Our results are contained in

**Theorem 1.** Suppose for $|z| < R$,

$$
\sum_{k=0}^{\infty} a_k z^k < \infty; \quad \sum_{k=0}^{\infty} b_k z^k < \infty.
$$

(2.1)

Then

$$
F(z, \sigma) = \sum_{k=0}^{\infty} a_k \left(\frac{\sigma}{k!}\right)^k \left(\frac{z}{\sigma}\right)^k;
$$

$$
G(z, \nu, \lambda) = \sum_{k=0}^{\infty} b_k \left(\frac{-\nu}{k!k!}\right)^k \left(\frac{z}{\nu(\nu + \lambda)}\right)^k,
$$

(2.2)

converge for $|z| < |\sigma| R$; $|z| < |\nu(\nu + \lambda)| R$, and can be rearranged in descending powers of $\sigma$; $\nu(\nu + \lambda)$, to yield the analytic confluent expansions,

$$
F(z, \sigma) = \sum_{j=0}^{\infty} g_j(z) \sigma^{-j}, \quad |z/\sigma| < R;
$$

$$
G(z, \nu, \lambda) = \sum_{j=0}^{\infty} h_j(z, \lambda) \left[-\nu(\nu + \lambda)\right]^{-j}, \quad \left|\frac{z}{\nu(\nu + \lambda)}\right| < R,
$$

(2.3) (2.4)

in which the $g_j(z)$ are entire functions of $z$ given explicitly by (2.7), and the $h_j(z, \lambda)$ are polynomials in $\lambda$ of degree $j$, whose coefficients are entire functions of $z$, which are given implicitly by (2.10) and (2.13). For $j \geq 1$, $g_j(z)$ and $h_j(z, \lambda)$ can be expressed in terms of the derivatives of $g_0(z)$ and $h_0(z, \lambda)$, respectively.

**Proof.** From the ratio test and (2.1), it follows that $F(z, \sigma); G(z, \nu, \lambda)$ converge for $|z| < |\sigma| R$; $|z| < |\nu(\nu + \lambda)| R$. First we prove (2.3). It follows from (1.6)
with $z = \sigma, \beta = 0, \text{and } \alpha = k$, together with certain generalized Bernoulli relationships in [4], that

\[
\frac{(\sigma)_k}{\sigma^k} = \sum_{j=0}^{k} \frac{(1-k)_j}{j!} B_j^{(k)}(0) \sigma^{-j}, \quad k = 0, 1, 2, \ldots.
\]

Clearly the coefficient of $\sigma^{-j}$ on the right of (2.5) is $\geq 0$, and $F(z, \sigma)$ is majorized by the series

\[
\sum_{k=0}^{\infty} |a_k| \left( \frac{1}{k!} \right) \left| \frac{z}{\sigma} \right|^k,
\]

which converges for $|z| < |\sigma| R$. Thus (2.5) can be substituted into the series definition of $F(z, \sigma)$ and the resulting series rearranged in powers of $\sigma^{-1}$. This leads to (2.3) with

\[
g_j(z) = \sum_{k=0}^{\infty} a_k \frac{(1-k)_j B_j^{(k)}(0)}{j! k!} z^k.
\]

To express $g_j(z)$ in terms of the derivatives of $g_0(z)$, we merely note that for fixed $j$, $(1-k) B_j^{(k)}(0)$ is a polynomial in $k$ of degree $2j$, and that it can be written in factorial powers of $k$, e.g. if $j = 3$,

\[
g_j(z) = \frac{1}{3} \sum_{k=0}^{\infty} a_k \left( \frac{1-k)_j B_j^{(k)}(0)}{j! k!} z^k.
\]

We now prove (2.4). For $\nu(\nu + \lambda) \neq 0, k$ and $j$ integers $\geq 0$, define the polynomials $C_{j,k}(\lambda)$ by

\[
\frac{(-1)^k (\nu)_{k} (\nu + \lambda)_{k}}{[\nu(\nu + \lambda)]^k} = (1 - k/\nu)^{-1} \prod_{j=0}^{k} (1 - j/\nu) \cdot (1 + k/\nu + \lambda)^{-1} \prod_{j=0}^{k} (1 + j/\nu + \lambda),
\]

\[
= \left( 1 + \frac{k(k + \lambda)}{(-\nu)(\nu + \lambda)} \right)^{-1} \prod_{j=0}^{k} \left( 1 + \frac{j(j + \lambda)}{(-\nu)(\nu + \lambda)} \right),
\]

\[
= \sum_{j=0}^{k} C_{j,k}(\lambda) [(-\nu)(\nu + \lambda)]^{-j}.
\]

From (2.10), it is easy to see that $C_{j,k}(\lambda)$ is a polynomial in $\lambda$ of order $j$, whose co-
coefficients are positive. Thus \( G(z, \nu, \lambda) \) is majorized by the series

\[
\sum_{k=0}^{\infty} \frac{|b_k| |z|^k}{k!} \sum_{j=0}^{k} C_{j,k}(|\lambda|) |\nu(\nu + \lambda)|^{-j}
\]

(2.11)

\[
= |b_0| + \sum_{k=1}^{\infty} \frac{|b_k| |z|^k}{k!} \prod_{j=0}^{j-1} \left( 1 + \frac{j(j + |\lambda|)}{|\nu(\nu + \lambda)|} \right),
\]

which in turn is majorized by

\[
|b_0| + \sum_{k=1}^{\infty} \frac{|b_k| |z|^k}{k!} \prod_{j=0}^{j-1} \frac{(|\nu|)_k(|\nu + |\lambda|)_k}{|\nu(\nu + \lambda)|} \left( \frac{z}{|\nu(\nu + \lambda)|} \right)^k, \quad |z| < |\nu(\nu + \lambda)| R.
\]

(2.12)

Thus \( G(z, \nu, \lambda) \) can be rearranged in descending powers of \( \nu(\nu + \lambda) \), and since \( \nu \) and \( \lambda \) were arbitrary,

\[
h_j(z, \lambda) = \sum_{k=0}^{\infty} \frac{b_k C_{j,k}(\lambda)}{k!} z^k,
\]

(2.13)

is a polynomial in \( \lambda \) of order \( j \), which converges for arbitrary \( z \). The final statement of the theorem follows as before. The \( C_{j,k}(\lambda) \) can be defined recursively. Multiplying (2.10) through by \( (\nu - k)(\nu + \lambda + k)[\nu(\nu + \lambda)]^{-1} \), one is led to the relation

\[
C_{j,k+1}(\lambda) - C_{j,k}(\lambda) = k(k + \lambda)C_{j-1,k}(\lambda); \quad k, j \geq 0.
\]

(2.14)

From (2.10) and (2.14), it follows that

\[
C_{j+1,k}(\lambda) = \sum_{m=0}^{k-1} m(m + \lambda)C_{j,m}(\lambda), \quad C_{0,m}(\lambda) = 1, \quad m \geq 0.
\]

(2.15)

Incorporating the same type of factorization as used to write \( g_j(z) \) in terms of the derivatives of \( g_0(z) \), we have by explicit computation from (2.15) for the first few \( C_{j,k}(\lambda) \),

\[
C_{0,k}(\lambda) = 1,
\]

\[
C_{1,k}(\lambda) = \frac{k(k - 1)}{6} [2(k - 2) + 3] + \frac{k(k - 1)}{2} \lambda,
\]

\[
C_{2,k}(\lambda) = \frac{k(k - 1)(k - 2)}{360} [20(k - 3)(k - 4)(k - 5) + 204(k - 3)(k - 4) + 495(k - 3) + 240]
\]

(2.16)

\[
+ \frac{k(k - 1)(k - 2)}{6} [(k - 3)(k - 4) + 6(k - 4) + 6] \lambda
\]

\[
+ \frac{k(k - 1)(k - 2)}{24} [3(k - 3) + 8]\lambda^2.
\]
Substitution of (2.16) into (2.13) then yields for the first few \( h_j(z, \lambda) \),

\[
h(z) = h_0(z, \lambda) = \sum_{k=0}^{\infty} \left( b_k/k! \right) z^k,
\]

(2.17) \[ h_1(z, \lambda) = ((\lambda + 1)/2)z h^{(2)}(z) + (z^3/3) h^{(3)}(z), \]

\[
h_2(z, \lambda) = ((\lambda^2 + 3\lambda + 2)/3) z h^{(3)}(z) + ((\lambda^2 + 8\lambda + 11)/8) z^4 h^{(4)}(z)
\]

\[ + ((5\lambda + 17)/30) z^5 h^{(5)}(z) + (z^5/18) h^{(6)}(z). \]

This completes the proof of the theorem.

**Remark I.** The coefficients of the \( g^{(j)}(z) \) in (2.9) are independent of the identity of the function \( g_0(z) \), and thus can be deduced from the special case when \( g_0(z) = e^z \), i.e.

(2.18) \[ F(z, \sigma) = (1 - z/\sigma)^{-\sigma} = e^{z \exp \left( \sum_{j=2}^{\infty} \sigma z^j/j! \right)}. \]

Similar remarks apply to the \( h^{(j)}(z) \) in (2.17).

**Remark II.** The characterization of \( g_j(z); h_j(z, \lambda) \), given in (2.9); (2.17), is particularly convenient when working with generalized hypergeometric functions, since for \( m, p, q \), integers \( \geq 0, p \leq q + 1 \),

\[
\frac{d^m}{dz^m} \mathbf{F}_q^p \left( \alpha_1, \ldots, \alpha_p \middle| \beta_1, \ldots, \beta_q \middle| z \right)
\]

(2.19) \[ = \left( \prod_{j=1}^{p} (\alpha_j)_m \right) / \left( \prod_{j=1}^{q} (\beta_j)_m \right) \mathbf{F}_q^p \left( m + \alpha_1, \ldots, m + \alpha_p \middle| \beta_1, \ldots, \beta_q \middle| z \right). \]

**Remark III.** If \( b_k = k! a_k \), the functions \( F \) and \( G \) are related by a confluen limit, i.e.

(2.20) \[ \lim_{\lambda \to \infty} G(z, \nu, \lambda) = F(z, -\nu). \]

**Remark IV.** If \( -\sigma = \nu = n \), an integer \( \geq 0 \), then \( F(z, -n) \) and \( G(z, n, \lambda) \) are polynomials in \( z \) of degree \( n \). Moreover, if the hypothesis (2.1) of Theorem 1 is replaced by the weaker hypothesis that for \( |z| < R^* \),

\[
\sum_{k=0}^{\infty} \frac{a_k}{k!} z^k < \infty; \quad \sum_{k=0}^{\infty} \frac{b_k}{k!} z^k < \infty,
\]

then the \( g_j(z); h_j(z, \lambda), \) which are now only defined for \( |z| < R^* \), are the Poincaré coefficients of \( F(z, -n); G(z, n, \lambda) \) as \( n \to \infty \). As is indicated by Remark III and the proof of Theorem 1, the proof of this fact for \( F(z - n) \) is similar to, but simpler than, the proof of the corresponding fact for \( G(z, n, \lambda) \). The result for \( G(z, n, \lambda) \) follows readily from the following

**Lemma 1.** Suppose \( m \) an integer \( \geq 0 \), and \( k, n \) integers such that \( 1 \leq k \leq n \). Then for \( n \) sufficiently large, the \( C_{j,k}(\lambda) \) defined by (2.10) satisfy the inequality,

\[
(m!) \left| \frac{(-1)^k (n + \lambda)_k}{[n(n + \lambda)]^k} \left( \sum_{j=0}^{n-1} C_{j,k}(\lambda) \right) \right| \leq \varepsilon^{(n + \alpha)}
\]

(2.22) \[ \lambda = \alpha + i\beta; \quad \alpha, \beta \text{ real}; \quad \mu = \max \{0, -\alpha(3|\alpha| + 2|\beta|)/4\}. \]

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To prove (2.22), it is sufficient to notice that for \( n \) sufficiently large,

\[
\begin{align*}
\text{Max}_{0 \leq u \leq 1; 0 \leq t \leq n} \left| 1 - \frac{ut(t + \lambda)}{n(n + \lambda)} \right| \leq 1 + \frac{\mu}{n(n + \alpha)},
\end{align*}
\]

and hence by direct computation, that

\[
\begin{align*}
\text{Max}_{0 \leq u \leq 1} | H_k^{(m)}(-u(n(n + \lambda))^{-1}) | \leq k^{2m}(k + | \lambda |)^m(1 + \mu(n(n + \alpha))^{-1})^n \leq k^{2m}(k + | \lambda |)^m e^{\mu(n + \alpha)},
\end{align*}
\]

\[
\begin{align*}
H_k(x) = \prod_{j=0}^{k-1} [1 + j(j + \lambda)x].
\end{align*}
\]

Combining (2.24) with Taylor's theorem, one arrives at (2.22).

III. Asymptotic Confluent Expansions. Here we give two canonical examples of asymptotic confluent expansions. Our first result is contained in

**Theorem 2.** Suppose for \( |z| < R, \)

\[
\sum_{k=0}^{\infty} c_k z^k < \infty.
\]

Then

\[
T(z, \rho) = \sum_{k=0}^{\infty} \frac{c_k}{(\rho)_k} (\rho z)^k,
\]

converges for all \( z, \rho \neq 0, -1, \ldots, \) and possesses the asymptotic confluent expansion

\[
T(z, \rho) \sim \sum_{j=0}^{\infty} f_j(z)(-\rho)^{-j},
\]

\[
\rho \to \infty, \quad |\arg \rho| \leq \pi - \delta, \quad 0 < \delta \leq \pi/2, \quad |z\rho| \leq R;
\]

\[
r_{\rho} = 1 \quad \text{if} \quad |\arg \rho| \leq \pi/2,
\]

\[
= |\sin (\arg \rho)|^{-1} \quad \text{if} \quad \pi/2 \leq |\arg \rho| \leq \pi - \delta,
\]

in which the \( f_j(z) \) are functions analytic in \( |z| < R \) given explicitly by (3.8). Moreover, for \( j \geq 1, f_j(z) \) can be expressed in terms of the derivatives of \( f_0(z) \).

**Proof.** From the ratio test, it follows that \( T(z, \rho) \) converges under the stated conditions. Next we note that (1.6) with \( z = \rho, \alpha = 0, \beta = k \), reduces to

\[
\frac{\rho^k}{(\rho)_k} = \sum_{j=0}^{\infty} \beta_{j,k} \rho^{-j}, \quad \beta_{j,k} = (-1)^{j}(k)_j B_{j}^{(1-k)}(0).
\]

Before computing the Poincaré coefficients of \( T(z, \rho) \), we need the following

**Lemma 2.** Suppose \( n \) an integer \( \geq 0, \) and \( k \) an integer \( \geq 1. \) Then for \( \rho \) sufficiently large, the \( \beta_{j,k} \) defined by (3.4) satisfy the inequality

\[
\begin{align*}
\left| \frac{\rho^k}{(\rho)_k} - \sum_{j=0}^{n-1} \beta_{j,k} \rho^{-j} \right| |\rho|^n(\rho)^{1-n-k}k^{-2n} \leq 1, \\
|\arg \rho| \leq \pi - \delta, \quad 0 < \delta \leq \pi/2.
\end{align*}
\]
Proof. If \(|\arg \rho| \leq \pi - \delta\) and \(ju \geq 0\), then
\[
(3.6) \quad 1 + jup^{-1} |^{-1} \leq r_\rho.
\]
It follows by an induction proof on \(n\), that
\[
\text{Max}_{0 \leq u \leq 1} |F_k^{(n)}(up^{-1})| \leq (n!)(r_\rho)^{k+n-1}k^2n, \quad n = 0, 1, \ldots ,
\]
\[
(3.7) \quad F_k(x) = \prod_{j=0}^{k-1} (1 + jx)^{-1}, \quad k = 1, 2, \ldots .
\]
Eq. (3.7) combined with Taylor's theorem, yields (3.5) and completes the lemma.

Now set
\[
(3.8) \quad f_j(z) = \sum_{k=0}^{\infty} c_k(-1)^j \beta_{j,k} z^k = \sum_{k=0}^{\infty} c_k \frac{(k)_j B_j^{(1-k)}(0)}{j!} z^k, \quad j = 0, 1, \ldots .
\]
Clearly, for fixed \(j\), the coefficient of \(cz^k\) in (3.8) is a polynomial in \(k\) of degree \((2j)\). Thus the functions \(f_j(z)\) are analytic in \(|z| < R\). We now show that the \((-1)^j f_j(z)\)
are the Poincaré coefficients of \(T(z, \rho)\) at \(\rho = \infty\), i.e. for \(\arg \rho\) fixed, \(|\arg \rho| \leq \pi - \delta\),
\[
(3.9) \quad \lim_{\rho \to \infty} (-\rho)^n \left[T(z, \rho) - \sum_{j=0}^{n-1} f_j(z)(-\rho)^{-j}\right]
= \frac{c_0}{\Gamma(1 - n)} + \sum_{k=1}^{\infty} c_k z^k \lim_{\rho \to \infty} \left\{\frac{\rho^k}{(\rho)^k} - \sum_{j=0}^{n-1} \beta_{j,k} \rho^{-j}\right\},
= \frac{c_0}{\Gamma(1 - n)} + \sum_{k=1}^{\infty} c_k \beta_{n,k} (-1)^n \rho^k,
= f_n(z).
\]
The interchange of limit processes in (3.9) follows from the fact that, in view of the
lemma, the original series in (3.9) is majorized by the series
\[
(3.10) \quad \frac{|c_0|}{\Gamma(1 - n)} + (r_\rho)^n \sum_{k=1}^{\infty} k^{2n} |c_k| |\rho r_\rho|^{k-1},
\]
which converges for \(|\rho r_\rho| < R\). Note that since \(r_\rho\) is a function of \(\arg \rho\) only, the
convergence is uniform in \(\rho\) on the ray \(t \exp (i \arg \rho), \rho | \leq t \leq \infty, \arg | \leq \pi - \delta\).
Finally, the representation of \(f_j(z)\) in terms of the derivatives of \(f_0(z)\) follows as in
Theorem 1, and the first few are,
\[
(3.11) \quad f(z) = f_0(z) = \sum_{k=0}^{\infty} c_k z^k, \quad f_1(z) = (z^2/2)f^{(2)}(z),
\]
\[
f_2(z) = (z^2/2)f^{(0)}(z) + (2z^3/3)f^{(3)}(z) + (z^4/8)f^{(4)}(z),
\]
which completes the proof of the theorem.

As a final canonical example of a confluent situation, we prove the following

Theorem 3. Suppose for \(|z| < R\),
\[
(3.12) \quad v(z) = \sum_{k=0}^{\infty} d_k z^k < \infty.
\]
Then
\begin{equation}
S(z, \sigma, a, b) = \sum_{k=0}^{\infty} d_k \frac{(\sigma + a)_k}{(\sigma + b)_k} z^k,
\end{equation}
converges for \(|z| < R\), \(\sigma + b \neq 0, -1, \ldots\), and can be rearranged in the region \(|z| < R/2\), to yield the expansion
\begin{equation}
S(z, \sigma, a, b) = \sum_{j=0}^{\infty} \frac{(b - a)_j (-z)^j}{(\sigma + b)_j j!} \psi^{(j)}(z).
\end{equation}

If \(a\) and \(b\) are bounded quantities (3.14) holds asymptotically in the larger region \(|z| < R\), i.e., if \(n\) is an integer > 0,
\begin{equation}
S(z, \sigma, a, b) = \sum_{j=0}^{n-1} \frac{(b - a)_j (-z)^j}{(\sigma + b)_j j!} \psi^{(j)}(z) + O(\sigma^{-n}),
\end{equation}
\(\sigma \to \infty\), \(|\arg (\sigma + b)| \leq \pi - \delta\), \(\delta > 0\), \(|z| < R\).

If \([(\sigma + b)]^{-1}\) is expanded in powers of \(\sigma^{-1}\) (3.15) can be written as an asymptotic confluent expansion in \(\sigma^{-1}\).

**Proof.** From the ratio test, it follows that \(S(z, \sigma, a, b)\) converges under the stated conditions. One sees from Gauss's formula for a \(\mathbf{2F1}\) of unit argument, [1], that if \((\sigma + b)_j, j = 1, 2, \ldots\),
\begin{equation}
\mathbf{2F1}\left(\begin{array}{c} k \cr 0, b - a \end{array} \sigma + b \right) = \frac{1}{(\sigma + b)_k}
\end{equation}
Assume that \(|z| < z_0 < R\). Then \(d_k = O(z_0^{-k})\) uniformly in \(k\), as \(k \to \infty\). Thus the right-hand side of (3.14) up to a multiplicative constant is majorized by
\begin{equation}
\sum_{j=0}^{k} \frac{(b - a)_j (-z)^j}{(\sigma + b)_j j!} \frac{1}{(\sigma + b)_j} \left(\frac{z}{z_0}\right)^{k-j}
\end{equation}
which converges for \(|z| < |z_0|/2\). Thus the right-hand side of (3.14) can be arranged in powers of \(z\), establishing (3.14). To prove the asymptotic expansion (3.15), we merely remark that the same methods used in Theorem 3 can be used to establish the existence of a Poincaré asymptotic expansion of \(S(z, \sigma, a, b)\) in powers of \(\sigma^{-1}\), under the stated conditions of the theorem. In the common region \(|z| < R/2\), both the asymptotic expansion in \(\sigma^{-1}\) and (3.15) must agree when \([(\sigma + b)]^{-1}\) is expanded in powers of \(\sigma^{-1}\). This is sufficient to identify the Poincaré coefficients, and establish (3.15). This completes the proof of the theorem.

**Remark V.** The region \(|z| < R/2\) is, in general, the largest circular region in which the right-hand side of (3.14) can converge. This follows from the special case of Theorem 3, known as Euler's formula, see [1],
\begin{equation}
\mathbf{2F1}\left(\begin{array}{c} \sigma + a, \alpha \cr \sigma + b \end{array} \mid z\right) = \sum_{j=0}^{\infty} \frac{(b - a)_j (-z)^j}{(\sigma + b)_j j!} \frac{d^j}{dz^j} \left\{(1 - z)^{-\alpha}\right\}
\end{equation}

\(\sigma \to \infty\), \(|\arg (\sigma + b)| \leq \pi - \delta\), \(\delta > 0\), \(|z| < R\).
In this example, $R = 1$, but the right-hand side of (3.18) converges only for $\text{Re}(z) < \frac{1}{2}$. Also note in this example that the right-hand side of (3.18) analytically continues the left-hand side of (3.18) outside its original circle of convergence, i.e. the unit circle.

Remark VI. In the special case $S(z, \sigma, a, b)$ is a hypergeometric series, (3.14) yields a proof of the fact that whenever a convergent hypergeometric series has a numerator parameter differing from a denominator parameter by a positive integer $m$, that the hypergeometric series can be written as the sum of $m$ hypergeometric series of lower order. Although there are many examples of such formulae in the literature, this result seems never to have been proved in general.

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