Rational Approximations to the Incomplete Elliptic Integrals of the First and Second Kinds*

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In this note we derive rational approximations (in Eqs. (20) and (21) below) to the integrals

\[ F(\varphi, k) = \int_{0}^{\varphi} (1 - k^2 \sin^2 t)^{-1/2} dt, \]

and

\[ E(\varphi, k) = \int_{0}^{\varphi} (1 - k^2 \sin^2 t)^{1/2} dt, \]

where \( k^2 \) is real and \( 0 < \varphi < \pi/2 \), by obtaining the main diagonal Padé approximations to closely related functions. It is sufficient to consider the case \( 0 < k^2 < 1 \), for if \( k^2 > 1 \),

\[ F(\varphi, k) = k_1 F(\beta_1, k_1) \quad \text{and} \quad E(\varphi, k) = k_1[E(\beta_1, k_1) + (1 - k^2)^2 F(\beta_1, k_1)], \]

\( k_1 = 1/k \) and \( \beta_1 = \arcsin(k \sin \varphi) \),

while if \( k^2 < 0 \),

\[ F(\varphi, k) = (1 - k^2)^{1/2} F(\beta_2, k_2) \quad \text{and} \quad E(\varphi, k) = (1 - k^2)^{-1/2} E(\beta_2, k_2) - \frac{k_2^2 \sin \beta_2 \cos \beta_2}{(1 - k^2 \sin^2 \beta_2)^{1/2}}, \]

\( k_2 = |k|(1 - k^2)^{-1/2} \) and \( \beta_2 = \arcsin\left(\frac{1 - k^2}{1 - k^2 \sin^2 \varphi}\right)^{1/2} \sin \varphi \).

Define \( m = k^2 \) and

\[ a = \left[\frac{(2 - m)^2}{1 + m}\right]^{1/3} > 0, \quad b = \left[\frac{(1 - 2m)^3}{(m - 2)(m + 1)}\right]^{1/3}, \]

\[ c = \left[\frac{(1 + m)^2}{m - 2}\right]^{1/3} < 0, \quad x = c + \frac{a - c}{\sin^2 \varphi}, \]

\[ h = a\left[c + \frac{b(2m - 1)}{m - 2}\right] < 0, \quad g = 2m - 1, \quad s = 2\left[\frac{2 - m}{3a}\right]^{1/2}, \]

\[ r(x) = x^3 + hx + g, \quad v(x) = \frac{(x - c)^3(x - a)}{x - b}, \]

\[ I_1(x) = \int_{x}^{a} [r(t)]^{-1/2} dt \quad \text{and} \quad I_2(x) = \int_{x}^{a} [v(t)]^{-1/2} dt. \]

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Then \( a > b > c \) are the real roots of \( r(z) = 0 \) and it follows from [1] that

\[
F(\varphi, k) = s^{-1}I_1(x) \quad \text{and} \quad E(\varphi, k) = s^{-1}I_2(x).
\]

Set

\[
G_1(x) = [r(x)]^{1/2}I_1(x), \quad G_2(x) = \frac{[v(x)]^{1/2}}{2x} I_2(x).
\]

Then \( G_l(x) \ (l = 1, 2) \) satisfies the differential equation

\[
r(x)\gamma_l(x)G'_l(x) - \delta_l(x)G_l(x) + r(x) = 0,
\]

where

\[
\gamma_1(x) = 1, \quad \gamma_2(x) = 2x, \quad \delta_1(x) = \frac{1}{3}(3x^2 + h), \\
\delta_2(x) = x^3 - 2(a + 2b)x^2 + (ab - bc - 3ac)x + 2abc.
\]

For convenience, we make the transformations

\[
z = 1/x, \quad G_1(z) = x^{-1}\left[2 + x^2H_1(x)\right], \quad G_2(z) = H_2(z).
\]

Then (8) becomes

\[
\eta_l(x)H'_l(x) + \rho_l(x)H_l(x) + \xi_l(x) = 0, \quad l = 1, 2,
\]

where

\[
\eta_1(x) = x(1 + hx^2 + gx^3), \quad \eta_2(x) = 2\eta_1(x), \quad \rho_1(x) = \frac{5}{2} + \frac{3h}{2} x^2 + gx^3, \\
\rho_2(x) = 1 - 2(a + 2b)x + (ab - bc - 3ac)x^2 - 2gx^3, \quad \xi_1(x) = -2h - 3gx, \\
\xi_2(x) = -1 - hx^2 - gx^3, \quad H_1(0) = \frac{4h}{5} \quad \text{and} \quad H_2(0) = 1.
\]

Main diagonal Padé approximations for the solution to (10) are readily computed by using the results of [2]. For completeness we list the recurrence relations which determine the main diagonal Padé approximations to \( H_l(x), \ l = 1, 2 \). In the notation of [2], we have: for \( l = 1 \);

\[
y(x) = H_1(x), \\
y_0 = y(0) = 4h/5,
\]

\[
p_0 = p_2 = 0, \quad p_1 = 1, \quad p_2 = h, \quad p_4 = g, \\
q_0 = 5/2, \quad q_1 = 0, \quad q_2 = 3h/2, \quad q_3 = g, \\
s_0 = -2h, \quad s_1 = -3g, \quad s_2 = s_3 = 0,
\]

and for \( l = 2, y(x) = H_2(x), \)

\[
y_0 = y(0) = 1, \\
p_0 = p_2 = 0, \quad p_1 = 2, \quad p_2 = 2h, \quad p_4 = 2g, \\
q_0 = 1, \quad q_1 = -2(a + 2b), \quad q_2 = ab - bc - 3ac, \quad q_3 = -2g, \\
s_0 = -1, \quad s_1 = 0, \quad s_2 = -h, \quad s_3 = -g.
\]

Let
(13) \[ y_n = \frac{A_n}{B_n}, \quad A_n = \sum_{k=0}^{n} a_{n,k} x^k, \quad B_n = \sum_{k=0}^{n} b_{n,k} x^k \]

be the \( n \)th-order main diagonal Padé approximations to \( y(x) \). Then \( A_n \) and \( B_n \) satisfy

(14) \[ A_n = (1 + \beta_n x) A_{n-1} + \alpha_n x^2 A_{n-2} \]

The equations which determine \( \alpha_n \) and \( \beta_n \) are

(15) \[ \alpha_n = -\tau_{n-1,1} \left[ (-1)^n \alpha_{n-1,1} p_1 + \alpha_{n-1,2} u_1 + 2 \sum_{j=3}^{n} \alpha_{n-1,j} \tau_{j-2,1} \right]^{-1}, \]

and

\[ \beta_n = -\tau_{n-1,2} + \alpha_{n,2} u_2 + 2 \sum_{j=3}^{n} \alpha_{n,j} \left( \tau_{j-2,2} + \beta_{j-1} \tau_{j-2,1} \right) \]

\[ \times \left[ 2\tau_{n-1,1} + \alpha_{n,2} u_1 + 2 \sum_{j=3}^{n} \alpha_{n,j} \tau_{j-2,1} \right]^{-1}, \quad n = 2, 3, 4 \ldots, \]

where

\[ \tau_{n,k} = \tau_{n-1,k+2} + 2\beta_n \tau_{n-1,k+1} + \alpha_n^2 \tau_{n-2,k} + \beta_n^2 \tau_{n-1,k} + (-1)^n \alpha_{n,1} p_k + \alpha_{n,2} u_k + \alpha_{n,3} u_{k+1} \]

\[ + 2 \sum_{j=3}^{n} \alpha_{n,j} \left[ \tau_{j-2,k+2} + \beta_{j-1} \tau_{j-2,k+1} + \beta_n \beta_{j-1} \tau_{j-2,k} \right], \]

(16) \[ n = 2, 3, 4 \ldots, k = 1, 2, 3 \]

\[ u_k = 2y_0 q_k + s_k + (a_{1,1} + b_{1,1} y_0) q_{k-1} + 2 b_{1,1} s_{k-1} \]

\[ \alpha_{k,j} = \alpha_k \alpha_{k-1} \cdots \alpha_j, \quad \alpha_{k,k} = \alpha_k, \quad \alpha_{k-1,k} = 1 \quad \text{and} \quad \alpha_{k,j} = 0, \quad k < j - 1. \]

The starting values for computation are

\[ \tau_{0,k} = y_0 q_k + s_k, \]

\[ \tau_{1,k} = -a_{1,k+2} y_0 q_{k+2} + s_{k+2} + (a_{1,1} + b_{1,1} y_0) q_{k+1} \]

\[ + 2 b_{1,1} s_{k+1} + a_{1,1} b_{1,1} q_k + b_{1,1} s_k, \quad k = 1, 2, 3 \]

for \( l = 1, \)

\[ \alpha_1 = -6g/7, \quad \beta_1 = b_{1,1} \]

\[ a_{1,1} = 6g/7 + 56h^2/225g, \quad b_{1,1} = 14h^2/45g; \]

for \( l = 2, \)

\[ \alpha_1 = -2/3(a + 2b), \quad \beta_1 = b_{1,1}, \]

\[ a_{1,1} = 4a^2 + 16b^2 + 25ab - 9bc - 27ac - 9h \]

\[ 30(a + 2b), \quad b_{1,1} = 4a^2 - 16b^2 - 19ab + 3bc + 9ac - 3h \]

\[ 10(a + 2b). \]

In either case, we have
Thus, rational approximations to the incomplete elliptic integrals of the first
and second kind respectively are

\begin{equation}
F_n(\phi, k) = \frac{[\tau(x)]^{1/2}}{s} \left[ 2x + \frac{A_n(1/x)}{xB_n(1/x)} \right]
\end{equation}

and

\begin{equation}
E_n(\phi, k) = \frac{2x[v(x)]^{-1/2}}{s} \frac{A_n(1/x)}{B_n(1/x)}.
\end{equation}

In the special case, \( k^2 = m = \frac{1}{2} \), the approximation (20) does not apply. However, since \( g = 0 \) in this case, (20) becomes

\begin{equation}
\frac{t(1 + ht)H'_1(t) + \frac{1}{2}(5 + 3ht)H_1(t)}{H_1(0)} = 0, \quad H_1(0) = 4h/5, \quad t = x^2,
\end{equation}

and \( H_1(t) = (4h/5) \_2F_1(1, 3/4; 9/4; -ht) \) is the solution to (22). Padé approximations to this hypergeometric function together with an error analysis are available in [3].

Numerical results indicate rapid convergence of the approximations (20) and (21). These approximations are evidently insensitive to changes in \( k^2 \) and are very powerful for \( \phi < \pi/3 \). They weaken as \( \phi \) approaches \( \pi/2 \); however, the Landen transformations

\begin{align*}
F(\phi, k) &= \frac{2}{1 + k} F(\phi_1, k_1), \\
E(\phi, k) &= (1 + k)E(\phi_1, k_1) + (1 - k)F(\phi_1, k_1) - k \sin \phi,
\end{align*}

where

\begin{equation}
k_1 = 2\sqrt{k/(1 + k)} \quad \text{and} \quad \phi_1 = \frac{1}{2} \phi + \frac{1}{2} \arcsin(k \sin \phi),
\end{equation}

should reduce \( \phi \) to the desirable range in all but the extreme cases. For example, if \( k = \frac{1}{2} \) and \( \phi = \pi/2 \) we have

\begin{equation}
F(\frac{1}{2}, \pi/2) = \frac{1}{3}F(2\sqrt{2}/3, \pi/3).
\end{equation}

The approximations \( \frac{1}{3}F_n(2\sqrt{2}/3, \pi/3) \) are listed in Table I.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
\textbf{n} & \textbf{\( \frac{1}{3}F_n \)} \\
\hline
\hline
4 & 1.68579 & 32446 \\
6 & 1.68575 & 05579 \\
8 & 1.68575 & 03557 \\
10 & 1.68575 & 03548 \\
12 & 1.68575 & 03548 \\
\hline
\end{tabular}
\caption{Table I}
\end{table}

The true value is 1.68575 \ 03548.

We present in Table II a tabulation of \( \varepsilon_n = |F(\phi, k) - F_n(\phi, k)| \) for a number of values of \( n, \phi \) and \( k \). The behavior of the error involved in approximating \( E(\phi, k) \)
by $E_n(\phi, k)$ is almost identical and so is omitted. In both tables $\epsilon_n < 1.0 \times 10^{-8}$ for $\phi \leq 30^\circ$ and $n \geq 4$ ($k$ arbitrary) so that these values are not listed. No entry in the table signifies an error less than $1.0 \times 10^{-8}$.

$$k^2 = .75$$

<table>
<thead>
<tr>
<th>$\phi$ \ $n$</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$60^\circ$</td>
<td>1.92 ($-3$)</td>
<td>4.7 ($-7$)</td>
<td>5.44 ($-3$)</td>
<td>1.13 ($-3$)</td>
<td>8.0 ($-7$)</td>
</tr>
<tr>
<td>$80^\circ$</td>
<td>1.51 ($-1$)</td>
<td>2.55 ($-2$)</td>
<td>5.44 ($-3$)</td>
<td>1.13 ($-3$)</td>
<td>8.0 ($-7$)</td>
</tr>
</tbody>
</table>

* The number in parentheses indicates the power of ten by which the tabular entry is to be multiplied.

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