A Steady State Phase Change Problem*

By A. Solomon

In a previous paper [4] the solution to the Stefan problem for a one-dimensional semi-infinite slab with constant boundary and initial conditions was shown to be given by the limit of solutions to a nonlinear parabolic equation for the “specific internal energy.” In this paper we obtain the same result for the Stefan problem in a bounded two- or three-dimensional domain, with constant boundary conditions. This result further justifies the application of the methods of [3], [4] to the Stefan problem in higher dimensions.

In Section 1 the problem to be solved is stated, and a simple solution given. In Sections 2, 3 this solution is shown to be obtainable from a limit of solutions to a related problem for the specific internal energy, as well as a solution to a related problem in the calculus of variations.

1. Notation and Statement of the Problem. Let £ be a bounded region of the $x, y$ plane having a smooth boundary $\Gamma$ and consisting of material which undergoes a change of phase, from Phase “I” to Phase “II,” at the critical temperature $T_c$ (see Fig. 1); our results apply as well for a three-dimensional region. (Phases I and II can represent “frozen” and “melted” states of the material.) Let $H$ be the latent heat of the material which is lost in the transition from Phase II to Phase I, $c_1, K_1$ and $c_2, K_2$ the specific heat and conductivity of Phase I and Phase II material, respectively, and $\kappa_i = K_i/c_i \rho$, $i = 1, 2$, where $\rho$ is the density of Phase I and II material, which we assume to be the same.

Suppose that the temperature $T$ of the boundary $\Gamma$ is given by the function

$$T(\sigma) = T^0(\sigma)$$

(with $\sigma$ the arc length on $\Gamma$) and maintained at this temperature for all time. $T^0$ is to be a bounded and piecewise continuous function of $\sigma$ assuming values above and below $T_c$. Then the steady state temperature $T(x, y)$ at points $(x, y)$ of £ is har-
monic throughout $\mathcal{L}$ except across certain curves $C$ in $\mathcal{L}$ marking the interface between Phase I and II materials. At $C$,

$$T = T_c,$$

(1b)

$$K_1|\text{grad } T^-| = K_2|\text{grad } T^+|,$$

(1c)

where $T^-$, $T^+$ denote the limiting temperatures at $C$ from within the Phase I and Phase II regions of $\mathcal{L}$, respectively.

We wish to determine a function $T(x, y)$ and curves $C$, which satisfy (1a, b, c) for given $T^0$. This can easily be done by a simple nonlinear change of variables. For let

$$K(\beta) = \begin{cases} 
K_1, & \text{for } \beta \leq T_c, \\
K_2, & \text{for } \beta > T_c,
\end{cases}$$

(2a)

and

$$U(x, y) = \int_0^{T(x,y)} K(\beta) d\beta ;$$

(2b)

then

$$U_{xx} + U_{yy} = 0$$

(2c)

in the Phase I and II regions, while at $C |\text{grad } U^+| = |\text{grad } U^-|$, with $U^-$, $U^+$ the limiting values of $U$ on $C$ from within the Phase I and II regions, respectively. Thus $U$ can be considered harmonic throughout $\mathcal{L}$. On $\Gamma$,

$$U(\sigma) = U^0(\sigma) = \int_0^{T^*(\sigma)} K(\beta) d\beta .$$

(2d)

A harmonic function $U(x, y)$ satisfying (2c, d) exists and can be found using well-known methods of potential theory (see [1]). Since the function $K$ of (2a) never vanishes, one can solve (2b) for the function $T(x, y)$ in terms of $U(x, y)$, which obeys (1a, b, c); moreover the interface curve $C$ on which $T = T_c$ is simply the equipotential curve for $U$ on which

$$U = \int_0^{T_c} K(\beta) d\beta .$$

2. A Related Problem for "Energy." Define $T$ and a function $\kappa$ as functions of a new variable $\epsilon$ by

$$T(\epsilon) = \begin{cases} 
T_c + (\epsilon - H)/c_1, & \text{for } \epsilon < H; \\
T_c, & \text{for } H \leq \epsilon \leq 2H; \\
T_c + (\epsilon - 2H)/c_2, & \text{for } \epsilon > 2H;
\end{cases}$$

(3)

$$\kappa(\epsilon) = \begin{cases} 
\kappa_1, & \text{for } \epsilon \leq H; \\
\phi_1(\epsilon), & \text{for } H \leq \epsilon \leq H + \epsilon; \\
\delta, & \text{for } H + \epsilon \leq \epsilon \leq 2H - \epsilon; \\
\phi_2(\epsilon), & \text{for } 2H - \epsilon \leq \epsilon \leq 2H; \\
\kappa_2, & \text{for } 2H \leq \epsilon,
\end{cases}$$

(4)
where $\epsilon, \delta$ are any (small) positive numbers, and $\phi_1, \phi_2$ are any smooth monotonic functions such that $\kappa(\epsilon), \kappa'(\epsilon)$ are continuous. Let $E^0(\sigma)$ be defined on $\Gamma$ in such a way that $T(E^0(\sigma)) = T^0(\sigma)$ (by (3)). Consider the boundary value problem

\begin{align}
(5a) & \quad (\kappa(\epsilon)e_x)_x + (\kappa(\epsilon)e_y)_y = 0 \quad \text{on} \quad \mathcal{L} ; \\
(5b) & \quad e = E^0 \quad \text{on} \quad \Gamma .
\end{align}

We claim that an analytic solution of (5a, b) exists, which as $\epsilon, \delta$ tend to zero, converges to a function yielding by (3) a piecewise harmonic function $T$ obeying (1a, b, c).

Let

\begin{equation}
F(\epsilon) = \int_0^\epsilon \kappa(\beta)d\beta ;
\end{equation}

then as a function of $x, y, F$ obeys (by (5a, b))

\begin{align}
(6a) & \quad F_{xx} + F_{yy} = 0 \quad \text{on} \quad \mathcal{L} , \\
(6b) & \quad F(\sigma) = \int_0^{E^0(\sigma)} \kappa(\beta)d\beta \quad \text{on} \quad \Gamma .
\end{align}

Under the assumptions on $E^0, \kappa$, such a function $F$ exists, and since $F'(\epsilon) = \kappa(\epsilon) \neq 0$, $\epsilon$ and $T$ may be found by (6), (3).

The equipotential curves for $F$ on which $\epsilon, F$ are constant, are Jordan arcs joining points of $\Gamma$. Let $\mathcal{L}^-, \mathcal{L}^0, \mathcal{L}^+$ be the subsets of $\mathcal{L}$ in which $\epsilon < H, H < \epsilon < 2H, \epsilon > 2H$, respectively. These regions are bounded by smooth Jordan arcs or sets of curves $C^H, C^{2H}$ in $\mathcal{L}$ on which $\epsilon = H, F = \kappa_1H$, and $\epsilon = 2H, F = \kappa_1H + \int_H^{2H} \kappa(\beta)d\beta$, respectively. The regions clearly depend on $\epsilon, \delta$.

Let $\epsilon$ tend to zero, with $\kappa(\epsilon)$ converging in a decreasing manner to the piecewise constant function

\begin{equation}
\kappa(\epsilon) = \begin{cases} 
k_1, & \text{for} \quad \epsilon \leq H ; \\
\delta, & \text{for} \quad H < \epsilon < 2H ; \\
k_2, & \text{for} \quad \epsilon \geq 2H ;
\end{cases}
\end{equation}

by Dini's theorem $F$ converges uniformly on $\Gamma$ to a continuous function given by (6b) (see [2, p. 106]). Consequently, as $\epsilon$ tends to 0, $F$ converges uniformly on $\mathcal{L} + \Gamma$ to a harmonic function satisfying (6a, b) with $\kappa$ defined by (7). By Harnack's theorem the domains $\mathcal{L}^+, \mathcal{L}^0, \mathcal{L}^-$ and curves $C^H, C^{2H}$ converge to domains $\mathcal{L}^+, \mathcal{L}^0, \mathcal{L}^-$ and smooth Jordan arcs $C^H, C^{2H}$ as above. From (6),

\begin{align}
(6c) & \quad F = \kappa_1\epsilon \leq \kappa_1H \quad \text{on} \quad \mathcal{L}^- ; \\
(6d) & \quad \kappa_1H \leq F = \kappa_1H + \delta (e - H) \leq H(\kappa_1 + \delta) \quad \text{on} \quad \mathcal{L}^0 ; \\
& \quad H(\kappa_1 + \delta) \leq F = H(\kappa_1 + \delta) + \kappa_2(e - 2H) \quad \text{on} \quad \mathcal{L}^+ ,
\end{align}

and

\begin{equation}
F = \begin{cases} 
\kappa_1H \quad \text{on} \quad C^H ; \\
H(\kappa_1 + \delta) \quad \text{on} \quad C^{2H} .
\end{cases}
\end{equation}

From (6c),
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\[ |\text{grad} F| = \begin{cases} 
\kappa_1|\text{grad} e|, & \text{in } \mathcal{L}^-; \\
(\kappa_1 + \delta)|\text{grad} e|, & \text{in } \mathcal{L}^0; \\
\kappa_2|\text{grad} e|, & \text{in } \mathcal{L}^+. 
\end{cases} \]

Since \( F(x, y) \) is constant on \( C^H \) it increases as the point \((x, y)\) crosses \( C^H \) from \( \mathcal{L}^- \) to \( \mathcal{L}^0 \). Let \( \varepsilon^- \), \( \varepsilon^0 \) be the limiting values of \( e \) on \( C^H \) from within \( \mathcal{L}^- \) and \( \mathcal{L}^0 \) respectively. Since \( F \) is harmonic throughout \( \mathcal{L} \),

\[(9a) \quad |\text{grad} \varepsilon^0| (\kappa_1 + \delta) = \kappa_1|\text{grad} \varepsilon^-| \]
on \( C^H \) (Eq. 8). Similarly, letting \( \varepsilon^+, \varepsilon^0 \) be the limiting values of \( e \) on \( C^H, C^{2H} \), from \( \mathcal{L}^+, \mathcal{L}^0 \) respectively,

\[(9b) \quad |\text{grad} \varepsilon^+| \kappa_2 = (\kappa_1 + \delta)|\text{grad} \varepsilon^0| ; \]

thus by (6), \( e \) is harmonic within \( \mathcal{L}^0, \mathcal{L}^-, \mathcal{L}^+ \), continuous throughout \( \mathcal{L} \), and has a discontinuous gradient across \( C^H, C^{2H} \).

Let \( \delta \) tend to 0. By reasoning similar to that above, \( \kappa(e) \) converges to the step function

\[ \kappa(e) = \begin{cases} 
\kappa_1, & \text{for } e \leq H; \\
0, & \text{for } H < e < 2H; \\
\kappa_2, & \text{for } e \geq 2H, 
\end{cases} \]

and \( F \) converges uniformly on \( \mathcal{L}^+ \) and \( \Gamma \) to a harmonic function obeying (6a, b). Moreover by (5d), \( C^H, C^{2H} \) converge to a common curve \( C \), on which \( F = H\kappa_1 \), while \( \mathcal{L}^0 \) tends to the empty set, and \( \mathcal{L}^+, \mathcal{L}^- \) tend to sets \( \mathcal{L}^{11}, \mathcal{L}^1 \), on which \( F > \kappa_1H \) and \( F < \kappa_1H \) respectively. The functions \( e \) converge to a function harmonic on \( \mathcal{L}^{11} \) and \( \mathcal{L}^1 \), and

\[ F = \begin{cases} 
\kappa_1e, & \text{on } \mathcal{L}^1; \\
\kappa_1H + \kappa_2(e - 2H), & \text{on } \mathcal{L}^{11}. 
\end{cases} \]

For \( \varepsilon^+, \varepsilon^- \) the limiting values of \( e \) at \( C \) from within \( \mathcal{L}^{11}, \mathcal{L}^1 \), respectively,

\[ \varepsilon^- = H, \varepsilon^+ = 2H, \quad \text{on } C. \]

Moreover, since \( \text{grad} F \) is continuous over \( \mathcal{L} \),

\[ \kappa_1|\text{grad} \varepsilon^-| = \kappa_2|\text{grad} \varepsilon^+|, \quad \text{on } C \]

and \( e = E^0 \) on \( \Gamma \). Using (3), we now obtain a function \( T(x, y) \) which is harmonic on \( \mathcal{L}^1, \mathcal{L}^{11} \) and obeys (1a, b, c). \( T \) is the solution to the steady state problem.

Since in the limit for \( \varepsilon = \delta = 0 \), \( F \) (and thus \( T \)) is determined uniquely by the given boundary values of \( T \) on \( \Gamma \), \( T \) is uniquely determined.

3. A Related Variational Problem. As a harmonic function continuous on \( \Gamma \), \( F \) is under suitable conditions the solution to the problem of minimizing the Dirichlet integral over \( \mathcal{L} \) among all functions obeying (6b) (see [1]). This implies by (3), (6), that the solution \( T \) to (1a, b, c) is that function minimizing the integral

\[ I = \frac{1}{\rho^2} \int_{\mathcal{L}} (K(T))^2 (T_x^2 + T_y^2) dxdy \]
with $K(T)$ defined by (2a), among all piecewise smooth functions $T$ obeying (1a).

Courant Institute of Mathematical Sciences
New York University
New York, New York 10012