
Midpoint Quadrature Formulas

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A family of quadrature formulas for the interval (0, 1) can be constructed in the following manner: For any positive integer n, we partition (0, 1) into subintervals $I_1, I_2, \ldots, I_n$ ($I_1$ being the leftmost, $I_2$ adjacent to it, etc.) of lengths $a_1, a_2, \ldots, a_n$, respectively. Now let $x_k$ be the midpoint of $I_k$, for $k = 1, \ldots, n$, and take

$$a_1 f(x_1) + \cdots + a_n f(x_n)$$

as the approximation to $\int_0^1 f(x)dx$. The simplest of these rules is the “Euler’s” or “midpoint” rule

$$\int_0^1 f(x)dx = f(\frac{1}{2}).$$

We will refer to the members of this family as “midpoint quadrature formulas” and determine their properties. We first find their “degrees of precision”—that is, for any formula, the highest integer $p$ such that the formula is exact for all polynomials of degree $p$ or lower.

**Theorem 1.** The degree of precision of a midpoint quadrature formula is 1.

**Proof.** The formula is exact for constants, since necessarily $a_1 + a_2 + \cdots + a_n = 1$. To check the exactness of the formula for $f(x) = x$, we first note that

$$x_1 = \frac{a_1}{2}, \quad x_2 = a_1 + \frac{a_2}{2}, \quad \ldots, \quad x_n = a_1 + \cdots + a_{n-1} + \frac{a_n}{2}.$$

So for the integral $\int_0^1 x\,dx$, (1) gives us

$$a_1(a_1/2) + a_2(a_1 + a_2/2) + \cdots + a_n(a_1 + \cdots + a_{n-1} + a_n/2).$$

Received December 5, 1966.
But this is just
\[
\frac{1}{2}(a_1^2 + a_2^2 + \cdots + a_n^2 + 2a_1a_2 + 2a_1a_3 + \cdots + 2a_{n-1}a_n) ,
\]
or \(\frac{1}{2}(a_1 + \cdots + a_n)^2\), which is \(\frac{1}{2}\). Thus the degree of precision is at least one. To show it is no greater, we calculate error in integrating \(x^2/2\) by the rule:
\[
\int_0^1 \frac{x^2}{2} \, dx - \sum_{i=1}^n a_i \frac{x_i^2}{2} = \frac{1}{6} - \frac{1}{2} \sum_{i=1}^n a_i \left( a_1 + a_2 + \cdots + a_{i-1} + \frac{a_i}{2} \right)^2 .
\]
Multiplying out and collecting terms in the last sum, we obtain:
\[
\sum_i a_i x_i^2 = \frac{1}{4} \sum_i a_i^3 + \sum_{i \neq j} a_i a_j^2 + 2 \sum_{i \neq j \neq k} a_i a_j a_k ,
\]
where the indices of summation run from 1 to \(n\).

Now
\[
1 = (a_1 + \cdots + a_n)^3 = \sum_i a_i^3 + 3 \sum_{i \neq j} a_i a_j^2 + 6 \sum_{i \neq j \neq k} a_i a_j a_k ,
\]
so that
\[
\frac{1}{3} - \sum_i a_i x_i^2 = \frac{1}{12} (a_1^3 + \cdots + a_n^3) .
\]
It follows that
\[
(3) \quad \int_0^1 \frac{x^2}{2} \, dx - \sum_i a_i \frac{x_i^2}{2} = \frac{1}{24} (a_1^3 + \cdots + a_n^3) > 0 ,
\]
which proves the theorem.

**Theorem 2.** The error of a midpoint quadrature formula, for an integrand with continuous second derivative, is given by
\[
(4) \quad \int_0^1 f(x)dx - \sum_i a_i f(x_i) = \frac{1}{24} (a_1^3 + \cdots + a_n^3)f''(\xi)
\]
for some \(\xi\) in \((0, 1)\).

**Proof.** By a general remainder theorem (see, e.g., [1]) the error may be written in the form
\[
(5) \quad \int_0^1 f''(t)K(t)dt
\]
where
\[
K(t) = \frac{(1 - t)^2}{2} - \sum_{i \geq 1} a_i (x_i - t) .
\]
To derive (4) from (5) it is sufficient to show that \(K(t)\) does not change sign in \((0, 1)\); for then we may write
\[
\int_0^1 f''(t)K(t)dt = f''(\xi) \int_0^1 K(t)dt ,
\]
and, taking \(f(x) = x^2/2\), we see from (3) that
We shall show that, in fact, \( K(t) \geq 0 \) for \( t \in [0, 1] \).

For \( t \) between \( x_k \) and \( x_{k+1} \),

\[
2K(t) = (1 - t)^2 - 2 \sum_{i=k+1}^{n} a_i(x_i - t) = (1 - t)^2 - 2 \sum_{i=k+1}^{n} a_i(1 - t) + 2 \sum_{i=k+1}^{n} a_i(1 - x_i) .
\]

Now, in fact

\[
2 \sum_{i=k+1}^{n} a_i(1 - x_i) = (a_{k+1} + a_{k+2} + \cdots + a_n)^2 .
\]

To prove this by induction, we need only show that

\[
2a_k(1 - x_k) = a_k^2 + 2a_k(a_{k+1} + \cdots + a_n) ,
\]

which follows directly from the fact that

\[
x_k = 1 - a_n - a_{n-1} - \cdots - a_{k+1} - a_k/2 .
\]

Therefore, in \([x_k, x_{k+1}]\),

\[
2K(t) = ((1 - t) - (a_{k+1} + \cdots + a_n))^2 \geq 0 ;
\]

and it can similarly be shown that \( K \) is nonnegative in \([0, x_1]\) and \([x_{n},1]\).

It is easy to see that, given \( n \), the coefficient \((a_1^3 + \cdots + a_n^3)/24\) in (4) is least when \( a_1 = a_2 = \cdots = a_n = 1/n \), so that for any \( n \), the “best” midpoint quadrature rule is simply the repeated Euler’s rule.

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