A family of quadrature formulas for the interval (0, 1) can be constructed in the following manner: For any positive integer \( n \), we partition (0, 1) into subintervals \( I_1, I_2, \ldots, I_n \) \( (I_1 \text{ being the leftmost, } I_2 \text{ adjacent to it, etc.}) \) of lengths \( a_1, a_2, \ldots, a_n \), respectively. Now let \( x_k \) be the midpoint of \( I_k \), for \( k = 1, \ldots, n \), and take

\[
a_1 f(x_1) + \cdots + a_n f(x_n)
\]

as the approximation to \( \int_0^1 f(x) \, dx \). The simplest of these rules is the “Euler’s” or “midpoint” rule

\[
\int_0^1 f(x) \, dx = f(\frac{1}{2}).
\]

We will refer to the members of this family as “midpoint quadrature formulas” and determine their properties. We first find their “degrees of precision”—that is, for any formula, the highest integer \( p \) such that the formula is exact for all polynomials of degree \( p \) or lower.

**Theorem 1.** The degree of precision of a midpoint quadrature formula is 1.

**Proof.** The formula is exact for constants, since necessarily \( a_1 + a_2 + \cdots + a_n = 1 \). To check the exactness of the formula for \( f(x) = x \), we first note that

\[
x_1 = \frac{a_1}{2}, \quad x_2 = a_1 + \frac{a_2}{2}, \quad \ldots, \quad x_n = a_1 + \cdots + a_{n-1} + \frac{a_n}{2}.
\]

So for the integral \( \int_0^1 x \, dx \), (1) gives us

\[
a_1(a_1/2) + a_2(a_1 + a_2/2) + \cdots + a_n(a_1 + \cdots + a_{n-1} + a_n/2).
\]
But this is just
\[ \frac{1}{2}(a_1^2 + a_2^2 + \cdots + a_n^2 + 2a_1a_2 + 2a_1a_3 + \cdots + 2a_{n-1}a_n) , \]
or \( \frac{1}{2}(a_1 + \cdots + a_n)^2 \), which is \( \frac{1}{2} \). Thus the degree of precision is at least one. To show it is no greater, we calculate error in integrating \( x^2/2 \) by the rule:
\[
\int_0^1 \frac{x^2}{2} \, dx - \sum_{i=1}^{n} a_i \frac{x^2}{2} = \frac{1}{6} - \frac{1}{2} \sum_{i=1}^{n} a_i \left( a_1 + a_2 + \cdots + a_{i-1} + \frac{a_i}{2} \right)^2 .
\]
Multiplying out and collecting terms in the last sum, we obtain:
\[
\sum_i a_i x_i^2 = \frac{1}{4} \sum_i a_i^3 + \sum_{i \neq j} a_i a_j^2 + 2 \sum_{i \neq j \neq k} a_i a_j a_k ,
\]
where the indices of summation run from 1 to \( n \).
Now
\[
1 = (a_1 + \cdots + a_n)^3 = \sum_i a_i^3 + 3 \sum_{i \neq j} a_i a_j^2 + 6 \sum_{i \neq j \neq k} a_i a_j a_k ,
\]
so that
\[
\frac{1}{6} - \sum_i a_i x_i^2 = \frac{1}{12} (a_1^3 + \cdots + a_n^3) .
\]
It follows that
\[
\int_0^1 \frac{x^2}{2} \, dx - \sum_i a_i \frac{x^2}{2} = \frac{1}{24} (a_1^3 + \cdots + a_n^3) > 0 ,
\]
which proves the theorem.

**Theorem 2.** The error of a midpoint quadrature formula, for an integrand with continuous second derivative, is given by
\[
\int_0^1 f(x)dx - \sum_i a_i f(x_i) = \frac{1}{24} (a_1^3 + \cdots + a_n^3)f''(\xi)
\]
for some \( \xi \) in \( (0, 1) \).

**Proof.** By a general remainder theorem (see, e.g., [1]) the error may be written in the form
\[
\int_0^1 f''(t)K(t)dt
\]
where
\[
K(t) = (1 - t)^2 - \sum_{i > t} a_i (x_i - t) .
\]
To derive (4) from (5) it is sufficient to show that \( K(t) \) does not change sign in \( (0, 1) \); for then we may write
\[
\int_0^1 f''(t)K(t)dt = f''(\xi) \int_0^1 K(t)dt ,
\]
and, taking \( f(x) = x^2/2 \), we see from (3) that
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\[ \int_0^1 K(t)dt = \frac{1}{24} (a_1^3 + \cdots + a_n^3). \]

We shall show that, in fact, \( K(t) \geq 0 \) for \( t \in [0, 1] \).

For \( t \) between \( x_k \) and \( x_{k+1} \),

\[
2K(t) = (1 - t)^2 - 2 \sum_{i=k+1}^{n} a_i(x_i - t)
= (1 - t)^2 - 2 \sum_{i=k+1}^{n} a_i(1 - t) + 2 \sum_{i=k+1}^{n} a_i(1 - x_i).
\]

Now, in fact

\[
2 \sum_{i=k+1}^{n} a_i(1 - x_i) = (a_{k+1} + a_{k+2} + \cdots + a_n)^2.
\]

To prove this by induction, we need only show that

\[ 2a_k(1 - x_k) = a_k^2 + 2a_k(a_{k+1} + \cdots + a_n), \]

which follows directly from the fact that

\[ x_k = 1 - a_n - a_{n-1} - \cdots - a_{k+1} - a_k/2. \]

Therefore, in \([x_k, x_{k+1}]\),

\[ 2K(t) = ((1 - t) - (a_{k+1} + \cdots + a_n))^2 \geq 0; \]

and it can similarly be shown that \( K \) is nonnegative in \([0, x_1]\) and \([x_n, 1]\).

It is easy to see that, given \( n \), the coefficient \((a_1^3 + \cdots + a_n^3)/24\) in (4) is least when \( a_1 = a_2 = \cdots = a_n = 1/n \), so that for any \( n \), the “best” midpoint quadrature rule is simply the repeated Euler’s rule.

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