TECHNICAL NOTES AND SHORT PAPERS

Explicit Gap Series at Cusps of $\Gamma(p)$

By A. O. L. Atkin

1. Let $\Gamma(1)$ be the full modular group consisting of all linear fractional transformations

$$\tau' = (a\tau + b)/(c\tau + d) \quad (a, b, c, d, \text{ rational integers and } ad - bc = 1) ,$$

and define subgroups as follows.

$\Gamma_0(n)$: $c \equiv 0 \pmod{n}$
$\Gamma_0^0(n)$: $b \equiv c \equiv 0 \pmod{n}$
Γ_n : $c \equiv 0,\ a \equiv d \equiv 1 \pmod{n}$
$\Gamma(n)$: $b \equiv c \equiv 0,\ a \equiv d \equiv 1 \pmod{n}$.

Any of these subgroups (say Γ) is a discontinuous group acting on H, the upper half-plane; we may regard H/Γ (compactified) as a Riemann surface S, genus g. With regard to a given point τ_0 of S we say that an integer $n \geqq 1$ is a *gap* if there exists no function with a pole of order n at τ_0 and regular elsewhere in S; otherwise n is a *nongap*. Weierstrass' Gap Theorem asserts that there are just g gaps n_i ($i = 1$ to g) and that these satisfy $1 \leqq n_i \leqq 2g - 1$; the sequence (n_i) is called the *gap series* at τ_0. For all but finitely many τ_0 the gap series is 1 to g; these exceptional τ_0 for which this is not true are called *Weierstrass points* of Γ (or, strictly, of S).

It is known (Schoeneberg [1]) that the cusp* $\tau = i\infty$ of $\Gamma(n)$ is a Weierstrass point for $n \geqq 7$ (and, since $\Gamma(n)$ is normal in $\Gamma(1)$, that all cusps are then Weierstrass points). For $\Gamma_0^0(n)$ it is known that $i\infty$ is a Weierstrass point for $n = 8$ and $n \geqq 10$ (Atkin [2]). Smart [3] has obtained much detailed information in general as to the gap series on $\Gamma(n)$, and proposed to me (private communication) the problem of its explicit determination in given cases (Lewittes [4] has done this for $n = 7$ to 10 and 12). For the case $n = p$, a prime, I give below a method which was successful for $p = 11, 13, 17, 19$, using the ICT Atlas I computer of the Science Research Council at Chilton; for $p \geqq 23$ the machine time required became excessive. The actual results appear in Table 1 at the end of Section 3.

2. We now suppose $p \geqq 5$ is a prime, and consider the functions

$$W_k(\tau) = \exp(12\pi i k^2\tau/p) \cdot \vartheta_1(4k\pi\tau|p\tau)/\vartheta_1(2k\pi\tau|p\tau) \tag{2.1}$$

studied by Fine [5], who established the transformation equation

$$W_k((a\tau + b)/(c\tau + d)) = \exp(12\pi i k^2 ab/p) \cdot W_{ak}(\tau) \text{ if } c \equiv 0 \pmod{p} . \tag{2.2}$$

We have also

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* That is, a parabolic vertex, which here must be $i\infty$ or a rational point on the real axis. See for example J. Lehner, *Discontinuous Groups and Automorphic Functions*, Math. Surveys, No. 8, Amer. Math. Soc., Providence, R. I., 1964, p. 129 and note 5a.

$$W_{k+p}(\tau) = W_{-k}(\tau) = W_k(\tau) \tag{2.3}$$

so that there are just $q = (p-1)/2$ distinct functions $W_k(\tau)$, which are clearly by (2.2) functions on $\Gamma(p)$. Further since all the functions $W_k(\tau)$ are finite and nonzero at the cusp $\tau = 0$ of $\Gamma_0(p)$, by Eq. (2.17) of Fine [5], the only cusps of $\Gamma(p)$ where $W_k(\tau)$ can have a pole or zero are those equivalent to $\tau = i\infty$ on $\Gamma_0(p)$, which are just those equivalent to $\tau = i\infty$ on $\Gamma_0^0(p)$. Now the coset representatives of $\Gamma(p)$ in $\Gamma_0^0(p)$ may be taken in the form $R_a(\tau) = (a\tau + bp)/(p\tau + d)$ $(a = 2 \text{ to } q)$, $R_1(\tau) = \tau$, so that we have using (2.2) the following result.

LEMMA 1. *Let $C_1 = i\infty$, $C_a = a/p$ $(a = 2$ to $q)$, be the inequivalent cusps of $\Gamma(p)$ which are equivalent to $i\infty$ on $\Gamma_0^0(p)$, and let $v_a(k)$ be the valence of $W_k(\tau)$ at C_a. Then, writing $v_1(k) = v(k)$, we have $v_a(k) = v(ak)$.*

We now consider

$$F(\tau) = \prod_{k=1}^{q} W_k^{b_k}(\tau) ,$$

where b_k $(k = 1 \text{ to } q)$ are any positive, negative, or zero integers. It follows from Lemma 1 and the previous results that $F(\tau)$ is a function on $\Gamma(p)$, regular and nonzero except possibly at the cusps C_a, whose valence at C_a is

$$V_a = \sum_{k=1}^{q} b_k v(ak) \qquad (a = 1 \text{ to } q) . \tag{2.4}$$

We shall say that a set $(V_1, V_2, \cdots, V_q)$ is a *cyclic set* if for some integral choice of b_k (2.4) is satisfied; if in addition $V_1 < 0$ and $V_a \geqq 0$ $(a = 2 \text{ to } q)$ we shall say that it is a *special set* and that $-V_1$ is a *value*. Then from the definition of Section 1 we have

LEMMA 2. *A value is a nongap for $\Gamma(p)$.*

The converse of course is not necessarily true in general, but it turns out to be true in all the cases we consider. Thus we get rather more than is asserted by the Gap Theorem; for nongaps we can obtain functions with a pole at $i\infty$ all of whose zeros are at other cusps.

3. Since the local variable at $i\infty$ on $\Gamma(p)$ is $x = \exp(2\pi i\tau/p)$, it follows from (2.1) that

$$\begin{aligned} v(k) &= 6k^2 - kp \qquad \text{if } 1 \leqq k < p/4 , \\ v(k) &= 6k^2 - 5kp + p^2 \qquad \text{if } p/4 < k \leqq q . \end{aligned} \tag{3.1}$$

We now let h be a primitive root of p and reorder $v(k)$ and b_k by writing

$$u(k) = v(\alpha_k) , \qquad c_k = b_{\alpha_k} \qquad (k = 1 \text{ to } q) , \tag{3.2}$$

where α_k is the least positive integer congruent to $\pm h^{k-1} \pmod p$, so that $u(k)$ $(k = 1 \text{ to } q)$ can be explicitly found from (3.1), and the inequalities necessary for a special set assume, using (2.3) and (2.4), the cyclic form

$$\begin{aligned} V_1 &= c_1u(1) + c_2u(2) + \cdots + c_qu(q) < 0 , \\ V_2 &= c_2u(1) + c_3u(2) + \cdots + c_1u(q) \geqq 0 , \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ V_q &= c_qu(1) + c_1u(2) + \cdots + c_{q-1}u(q) \geqq 0 , \end{aligned} \tag{3.3}$$

for some integral choice of c_k $(k = 1 \text{ to } q)$.

(We have also reordered, but not renamed, the V_a for $a > 1$.) We observe that the problem of satisfying (3.3) is a rather special case of integer programming, but in fact we proceeded as follows. It is easily seen from (3.1) and (3.3) that if any of the V_a is divisible by p then so are all the others; in this case $F(\tau)$ is in fact a function on Γ_p and not merely on $\Gamma(p)$, and if n is a value arising from a special set then n/p is a nongap for Γ_p at the cusp $i\infty$. Thus starting with the $u(k)$ we obtain cyclic sets with $V_a \equiv 0 \pmod p$; using these we obtain by a small random search special sets with $V_a \equiv 0 \pmod p$. Now using these as bases we perform further small random searches to satisfy (3.2); once we have obtained g values less than or equal to $2g$ the gap series is known, consisting of the g numbers less than $2g$ which are not values.

Since $\Phi(\tau) = \sum_{a=1}^{q} F(R_a\tau)$ is a function on $\Gamma_0{}^0(p)$ we see also that $-\min_a V_a$ is a nongap* for $\Gamma_0{}^0(p)$ at $i\infty$ for any cyclic set V_a, which turned out to be sufficient to identify the gap series at $i\infty$ for $\Gamma_0{}^0(p)$ in the cases given.

While the evidence of these results is insufficient to justify any formal conjecture, the natural question arises as to whether for all prime p one can obtain the gap series at $i\infty$ for $\Gamma(p)$, $\Gamma_0{}^0(p)$, Γ_p and $\Gamma_0(p)$ by the use of these rather special functions. A further problem is whether $i\infty$ is a Weierstrass point of Γ_p for $p \geqq 17$, which seems possible. As to $\Gamma_0(p)$, I have not found any value of p for which $i\infty$ is a Weierstrass point.

In the table below we give the genus of each subgroup and list the nongaps less than or equal to g and the gaps greater than g. Otherwise numbers less than or equal to g are gaps, and numbers greater than g are nongaps.

* Provided the leading terms do not cancel, for which a sufficient condition is that the minimum is attained an odd number of times.

TABLE 1

Group	*Genus*	*Nongaps*	*Gaps*
Γ_{11}	1	..	..
Γ_{13}	2	..	..
Γ_{17}	5	5	6
Γ_{19}	7	6	8
$\Gamma_0{}^0(11)$	6	5	11
$\Gamma_0{}^0(13)$	8	7	9
$\Gamma_0{}^0(17)$	17	12	20
$\Gamma_0{}^0(19)$	22	15, 22	23, 31
$\Gamma(11)$	26	19, 22, 24, 25	27, 30, 31, 36
$\Gamma(13)$	50	39, 42, 47	55, 60, 64
$\Gamma(17)$	133	85, 88, 110, 113, 115, 119, 123, 126, 127, 133	134, 135, 139, 143, 144, 149, 150, 157, 164, 178
$\Gamma(19)$	196	114, 147, 161, 162, 171, 174, 177, 179, 186, 189, 190, 191, 192, 194, 195	197, 198, 199, 200, 204, 205, 206, 213, 220, 226, 227, 229, 233, 234, 256

4. I am indebted to the referee for pointing out that the description in Section 3 of the computation as it appears to the machine is theoretically complete but unilluminating to the reader. Accordingly we give here a description of the process for the case $p = 11$ with some numerical details. We have

$$p = 11, \quad q = 5, \quad v(1) = -5, \quad v(2) = 2, \quad v(3) = 10, \quad v(4) = -3, \quad v(5) = -4\,,$$

from (3.1). Next 2 is a primitive root of 11, and we take $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 4$, $\alpha_4 = 3$, $\alpha_5 = 5$, giving by (3.2) the values

$$u(1) = -5, \quad u(2) = 2, \quad u(3) = -3, \quad u(4) = 10, \quad u(5) = -4\,.$$

Thus the inequalities of (3.3) become

$$\begin{aligned}
V_1 &= -5c_1 + 2c_2 - 3c_3 + 10c_4 - 4c_5 < 0\,,\\
V_2 &= 2c_1 - 3c_2 + 10c_3 - 4c_4 - 5c_5 \geqq 0\,,\\
V_3 &= -3c_1 + 10c_2 - 4c_3 - 5c_4 + 2c_5 \geqq 0\,,\\
V_4 &= 10c_1 - 4c_2 - 5c_3 + 2c_4 - 3c_5 \geqq 0\,,\\
V_5 &= -4c_1 - 5c_2 + 2c_3 - 3c_4 + 10c_5 \geqq 0\,.
\end{aligned}$$

We now exhibit choices of c_i ($i = 1$ to 5) giving cyclic sets V_i ($i = 1$ to 5), using the symbol C for cyclic sets and S for special sets.

c_1	c_2	c_3	c_4	c_5	V_1	V_2	V_3	V_4	V_5	
0	−1	0	−2	0	−22	11	0	0	11	(S1)
0	0	−1	0	−2	11	0	0	11	−22	(C1)
0	−2	−1	−4	−2	−33	22	0	11	0	(S2) = 2(S1) + (C1)
1	1	0	0	0	−3	−1	7	6	−9	(C2)
1	0	0	−2	0	−25	10	7	6	2	(S3) = (S1) + (C2)

. . . etc.

The special sets (S1) and (S2) with $V_i = 0 \pmod{11}$ are found first, giving values 22 and 33; then (C2) is a typical cyclic set with small V_i which added to (S1) gives a special set (S3) and a value 25. The random search consists of testing the addition to (S1) and (S2) of various cyclic sets similar to (C2); whenever $V_i \geqq 0$ ($i = 2$ to 5) we obtain a value $-V_1$.

The Atlas Computer Laboratory
Chilton, Didcot, England.

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