

# Powers of a Matrix of Special Type

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In this paper it is shown that if  $E^T E = I$ , i.e., if the columns of  $E$  are orthonormal, and  $K$  is a diagonal matrix with all terms positive, then

$$(1) \quad [I + E(K - I)E^T]^n = I + E(K^n - I)E^T$$

for any real  $n$ . Since given any matrix  $V$  and a positive definite matrix  $G$  it is possible to find an  $E$  and  $K$  as just described satisfying

$$(2) \quad VGV^T = E(K - I)E^T:$$

this provides a method for finding any power or root of matrices of the type

$$(3) \quad B = I + V G V^T.$$

This becomes particularly useful for work on high speed digital computers when  $G$  is very small. For suppose  $V$ —and hence also  $E$ —is  $n \times r$  with  $n \gg r$  and  $G$  is  $r \times r$ . Then keeping only  $E$  and, trivially, the diagonal of  $K$ , in fast-access storage and using only  $r$  core locations as working storage one can perform a rapid multiplication of an  $n \times 1$  vector by any of the matrices  $B, B^{1/2}, B^{-1/2}, B^{-1}$ , etc., with  $B$  as in (3).

Equation (1) can be easily proved from the identity\*

$$(4) \quad [I + E(K^p - I)E^T][I + E(K^q - I)E^T] = I + E(K^{p+q} - I)E^T$$

which can be established by merely multiplying out the terms on the left side and making use of the relation  $E^T E = I$ .

The conversion indicated in Eq. (2) can be accomplished by orthogonalizing the columns of  $V$  by elementary column operations [1]—the round-off error problem [2], not being significant when  $r$  is small—to obtain the matrix  $O$  such that

$$(5) \quad O^T O = I.$$

Let the product of the corresponding elementary matrices be the matrix  $R$ , i.e.,

$$(6) \quad O = VR \quad \text{or}$$

$$(7) \quad V = OR^{-1}.$$

Then let

$$(8) \quad H = R^{-1}GR^{-1T}.$$

Note that  $H$  is  $r \times r$  and symmetric positive definite. One then solves the small eigenproblem

$$(9) \quad HX = XL$$

for  $X$  and the diagonal matrix  $L$ . It follows that

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Received November 10, 1967.

\* The author is indebted to the referee for this equation.

$$(10) \quad X^T X = I,$$

$$(11) \quad X^T = X^{-1}.$$

Then, using Eqs. (7) through (11)

$$(12) \quad VGV^T = OHO^T = OXLX^TO^T = ELE^T$$

where

$$(13) \quad E = OX.$$

Furthermore  $E^TE = X^TO^TOX = X^TX = I$  using Eqs. (13), (5), and (10).

An alternative method, though somewhat longer but more stable in terms of rounding errors, is to perform a Cholesky factorization [3] of the positive definite matrix  $G$ , obtaining

$$(14) \quad G = UU^T.$$

Then form the matrix

$$(15) \quad C = U^TV^TVU$$

and solve the small  $r \times r$  eigenproblem

$$(16) \quad CY = YL$$

for  $Y$  and  $L$ . Since  $C$  is positive definite all terms of  $L$  are positive and one may set

$$(17) \quad E = VUYL^{-1/2}$$

so that

$$(18) \quad E^TE = L^{-1/2}Y^TU^TV^TVUYL^{-1/2} = L^{-1/2}Y^TCYL^{-1/2} = L^{-1/2}LL^{-1/2} = I$$

and

$$(19) \quad ELE^T = VUYL^{-1/2}LL^{-1/2}Y^TU^TV^T = VGV^T$$

as desired. Then let

$$(20) \quad K = I + L$$

for  $K$  as in Eq. (2).

Another identity related to (4), though not so general, for  $M$  and  $K$  diagonal matrices of nonzero terms and  $F$  such that  $F^TMF = I$ , is

$$(21) \quad [M + MF(K - I)F^TM][M^{-1} + F(K^{-1} - I)F^T] = I$$

which again can be proved by simply multiplying out the terms of the product.

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2. J. H. WILKINSON, *Rounding Errors in Algebraic Processes*, Prentice-Hall, Englewood Cliffs, N. J., 1963. MR 28 #4661.
3. A. RALSTON & H. S. WILF, *Mathematical Methods for Computers*, Vol. 2, Wiley, New York, 1967, p. 71.