On a Generalization of the Midpoint Rule*

By Franz Stetter

I. Introduction. A modified midpoint rule for the approximate calculation of weighted integrals $\int_a^b p(x)f(x)dx$, where $p(x) \geq 0$ is the weight function, has been recently proposed by Jagermann [1]. Although this formula reduces to the common midpoint rule in the particular case $p(x) = 1$, in the general case of arbitrary weight functions the error does not vanish for all polynomials $\alpha + \beta x$. The purpose of this paper is to generalize the midpoint rule such that the formula is exact for polynomials of first degree and arbitrary weight function $p(x) \geq 0$.

In view of practical calculations, the repeated midpoint rule is very useful because of its simplicity and small round-off error. Moreover, an error estimation does not require higher derivatives whose bounds are often not easy to obtain. For a comparison of the repeated midpoint rule to both Gaussian quadratures and “best” quadratures we refer to Stroud and Secrest [2].

II. Generalized Midpoint Rule. We assume that the weight function $p(x)$ does not identically vanish on any subinterval of $[a, b]$. Let

\begin{equation}
y = H(x) = \int_a^x p(t)dt, \quad H(b) = 1,
\end{equation}

and let the inverse function of $H$ (which exists because $H(x)$ is monotone increasing) be denoted by $L$:

\begin{equation}
x = L(y) = H^{-1}(y).
\end{equation}

For $i = 0, 1, \ldots, N - 1, (N \geq 1)$ we put

\begin{equation}
a_i = N \int_{i/N}^{(i+1)/N} L(y)dy = N \int_{x_i}^{x_{i+1}} tp(t)dt,
\end{equation}

where $x_i = L(i/N)$. We now define the generalized rule by:

\begin{equation}
\int_a^b p(x)f(x)dx = \frac{1}{N} \sum_{i=0}^{N-1} f(a_i) + R_N.
\end{equation}

Assuming $f \in C^2[a, b]$ the error $R_N$ can be expressed by

\begin{equation}
R_N = \frac{1}{2} \left( \int_a^b x^2p(x)dx - \frac{1}{N} \sum_{i=0}^{N-1} a_i^2 \right) f'''(\xi) = \frac{1}{2} C_N f'''(\xi), \quad a < \xi < b.
\end{equation}

Proof. Dividing $[a, b]$ into the subintervals $[x_i, x_{i+1}]$ we obtain for the error $R_N$

\begin{equation}
R_N = \sum_{i=0}^{N-1} \left\{ \int_{x_i}^{x_{i+1}} p(x) f(x)dx - \frac{1}{N} f(a_i) \right\}.
\end{equation}
By the Taylor series
\[ f(x) = f(a_i) + (x - a_i)f'(a_i) + \frac{1}{2}(x - a_i)^2f''(\xi_i) \]
and by (3) we get the expression
\[ R_N = \frac{1}{2} \sum_{i=0}^{N-1} \left\{ \int_{x_i}^{x_{i+1}} (x - a_i)^2 p(x) f''(\xi_i) \right\} \]
(6)
\[ = \frac{1}{2} \left( \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} (x - a_i)^2 p(x) dx \right) f''(\xi). \]

Furthermore, it follows from (3) that
\[ \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} (x - a_i)^2 p(x) dx = \sum_{i=0}^{N-1} \left\{ \int_{x_i}^{x_{i+1}} x^2 p(x) dx - \frac{2}{N} a_i^2 + \frac{1}{N} a_i \right\} \]
\[ = \int_{a}^{b} x^2 p(x) dx - \frac{1}{N} \sum_{i=0}^{N-1} a_i^2 . \]
(7)

(6) and (7) yield the bound (5).

\( C_N \) can also be interpreted as the integration error of the function \( f = x^2 \). It may be noted that Jägermann's modification of the midpoint rule is obtained if the integral \( N \int_{v_{1/N}}^{v_{1/N}} L(y)dy \) in (3) is approximated by the (ordinary) midpoint rule, i.e., by \( L((2i + 1)/2N) \).

III. Examples.
(a) For \( p(x) = 1 \) and \( a = 0, b = 1 \), we obtain \( a_i = (2i + 1)/2N \) and, from (5), \( C_N = 1/12N^2 \) in accordance with the common midpoint rule.

(b) Let \( p(x) = \pi^{-1}(1-x^2)^{-1/2} \) and \( a = -1, b = 1 \). From \( L(y) = -\cos \pi y \) it immediately follows that:
\[ a_i = -\frac{2N}{\pi} \sin \frac{\pi}{2N} \cos \frac{2i + 1}{2N} \pi \quad (i = 0, \ldots, N - 1) \]
and
\[ C_N = \frac{1}{2} - \frac{1}{N} \sum_{i=0}^{N-1} a_i^2 = \frac{1}{2} \quad \text{for } N = 1 \]
\[ = \frac{1}{2} - \frac{2N^2}{\pi^2} \sin \frac{\pi}{2N} \quad \text{for } N \geq 2 . \]

Obviously, \( C_N = O(N^{-2}) \).

(c) For the infinite interval \( a = 0, b = \infty \) and the weight function \( p(x) = e^{-x} \) we get from \( L(y) = -\log (1 - y) \):
\[ a_N = 1 + \log N , \]
\[ a_i = 1 + \log N - (N - i - 1) \log (N - i - 1) - (N - i) \log (N - i) \]
for \( i = 0, 1, \ldots, N - 2 \). Numerically computed values of \( C_N \)
show that $C_N$ goes to 0 with the order $O(N^{-1})$.

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