Error Bounds for Gauss-Chebyshev Quadrature*

By Franz Stetter

1. Introduction. For the error $E(f)$ of the numerical quadrature

$$E(f) = \int_a^b f(x) \, dx - \sum_{k=1}^N a_k f(x_k),$$

Davis [1] was the first to give bounds of the kind $\sigma \|f\|$ which do not involve derivatives of the function $f$, but $f$ is assumed to be analytic in a region containing the interval $[a, b]$. Since then such estimates have been developed in various directions, e.g., different norms of $f$, influence of the interval length, or optimal choice of the coefficients $a_k$ and $x_k$.

In this paper, we show that similar bounds can also be derived for quadrature rules based on suitable weight functions $w(x)$. We especially consider Gaussian rules with the weight $w(x) = (1 - x^2)^{-1/2}$ over the interval $[-1, 1]$:

$$R(f) = \int_{-1}^1 \frac{f(x)}{(1 - x^2)^{1/2}} \, dx - \frac{\pi}{N} \sum_{\nu=1}^N f \left( \cos \left( \frac{2\nu - 1}{2N} \pi \right) \right).$$

In this connection we also refer to Stenger [4] who gives general error developments.

2. Bounds. Let $f$ be analytic in $|z| \leq r$, $r > 1$. Applying the linear and continuous operator $R$ to the Cauchy integral of $f$

$$f(z) = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z - x} \, dz = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(re^{i\phi})}{1 - xre^{-i\phi}} \, d\phi,$$

we therefore immediately get by means of the Cauchy-Schwarz inequality:

$$|R(f)|^2 \leq \left\{ \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{|R(x^n)|^2}{r^{2n}} \right\} \left\{ \int_0^{2\pi} |f(re^{i\phi})|^2 \, d\phi \right\} = \sigma^2 \|f\|^2,$$

where $\sigma^2$ depends on $N$ and $r$. ($r$ is to be chosen so that $\sigma^2 \|f\|^2$ is minimal.) Now $R(x^n) = 0$ for $n = 0, 1, \cdots, 2N - 1$ (the rule (1.1) is exact for polynomials of degree less than $2N$) and $R(x^{2n+1}) = 0$ for $n = 0, 1, \cdots$ ((1.1) is symmetric). From

$$\int_{-1}^1 \frac{x^{2n}}{(1 - x^2)^{1/2}} \, dx = \frac{\pi}{2} \frac{1 \cdot 3 \cdots (2n - 1)}{2 \cdot 4 \cdots (2n)} = \frac{\pi}{2} \frac{(2n - 1)!!}{(2n)!!},$$

we obtain the expression

$$\sigma^2 = \frac{\pi}{2} \sum_{n=0}^{\infty} \left( \frac{(2n - 1)!!}{(2n)!!} \right) - \frac{1}{N} \sum_{\nu=1}^N \cos^{2n} \left( \frac{2\nu - 1}{2N} \pi \right) r^{-4n}.$$

(a) Case $N = 1$. Since $\cos \pi/2 = 0$ we get

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\[ \sigma^2 = \frac{\pi}{2} \sum_{n=1}^{\infty} \left( \frac{(2n-1)!!}{(2n)!!} \right)^2 r^{-4n} < \frac{\pi}{8} \frac{1}{r^4 - 1}, \]

and hence,

\[ |R(f)| \leq \frac{1}{2} \left( \frac{\pi}{2(r^4 - 1)} \right)^{1/2} \|f\|. \]

(b) Case \( N = 2 \). Now

\[ \sigma^2 = \frac{\pi}{2} \sum_{n=1}^{\infty} \left( \frac{(2n-1)!!}{(2n)!!} - \frac{1}{2^n} \right)^2 r^{-4n}. \]

From

\[ \text{Max}_n \left( \frac{(2n-1)!!}{(2n)!!} - \frac{1}{2^n} \right)^2 = \left( \frac{55}{256} \right)^2 \]

it follows that

\[ |R(f)| \leq \frac{55}{256 r^2} \left( \frac{\pi}{2(r^4 - 1)} \right)^{1/2} \|f\|. \]

(c) Case \( N = 3 \). \( |R(x^{2n})| \) assumes its maximum for \( n = 12 \); the value is

\[ 0.14006 \cdots < 1/7. \]

Hence

\[ |R(f)| < \frac{1}{7r^4} \left( \frac{\pi}{2(r^4 - 1)} \right)^{1/2} \|f\|. \]

(d) Case \( N \geq 4 \).

In the theory of numerical integration, it is shown that the error \( R(f) \) can be expressed by \( R(f) = af^{(2N)}(\xi) \) where \( a > 0 \) and \(-1 \leq \xi \leq 1\); hence (for \( n \geq N \)),

\[ 0 \leq R(x^{2n}) = \pi \left( \frac{(2n-1)!!}{(2n)!!} - \frac{1}{N} \sum_{r=1}^{N} \cos^2 \left( \frac{2\pi - 1}{2N} \right) \right) \]

\[ \leq \pi \frac{(2n-1)!!}{(2n)!!} \leq \pi \frac{(2N-1)!!}{(2N)!!}. \]

Thus, we get the general estimate

\[ |R(f)| \leq \frac{(2N-1)!!}{(2N)!! r^{2N-2} \left( \frac{\pi}{2(r^4 - 1)} \right)^{1/2} \|f\|. \]

The bound (2.10) is also valid for \( N = 1, 2, 3 \). Using Stirling's formula for \( n! \), we get from (2.10)

\[ |R(f)| \leq \frac{1.05}{r^{2N-2}(2N)^{1/2}} \left( \frac{1}{r^4 - 1} \right)^{1/2} \|f\| \]

for every \( N \geq 1 \).
3. Example. Let \( f(x) = x^6 \). Then the two-point rule \((N = 2)\) has the bound

\[
|R(f)| \leq \frac{55}{256} \frac{\pi r^4}{(r^4 - 1)^{1/2}}
\]

which is minimum when \( r^4 = 2 \). Thus, \( |R(f)| \leq 55\pi/128 \). The exact error is \( 3\pi/16 \).

4. Remarks. Similar results can be derived by using, instead of the norm used in this paper, polynomials orthogonal over the region \(|z| < r\) or orthogonal over (or on) the ellipse whose foci are \( \pm 1 \). For details we refer to Davis [2].

As mentioned by Hämmerlin [3] and Stenger [4], the norm \( ||f|| \) can be replaced by \( ||f - P_{2N-1}|| \) where \( P_{2N-1} \) is a polynomial of degree \( 2N - 1 \).