The Generalized G-Transform

By H. L. Gray and T. A. Atchison†

1. Introduction. In a recent paper [1], H. L. Gray and T. A. Atchison have considered a class of nonlinear transformations which can be used in the evaluation of improper integrals. In addition, Gray [2] has introduced some limiting forms associated with these transformations and Gray and Schucany [3] and [4] have studied their application as well as introduced some additional transformations.

All of the transformations discussed in the papers mentioned above are essentially of the same nature. In fact, each one is actually obtainable from the same ideas that produced the initial transformation, i.e., the G-transform. The purpose of this paper is to establish the ideas involved in defining the G-transform and to show how they can be used in such a manner as to establish a general class of transformations. Moreover, it is shown that the G-transform can be derived from this class. In addition, a new transformation termed the "B-transform" is established which is found to be useful on those integrals for which the G-transform is not well suited.

2. The Generalized G-Transform. The motivation for the transform which we wish to consider and which we will refer to as the generalized G-transform is as follows:

Consider any two improper integrals of the first kind defined by $\lim_{t\to\infty} F(t)$ and $\lim_{t\to\infty} H(t)$, where

$$F(t) = \int_a^t f(x)dx \quad \text{and} \quad H(t) = \int_a^t h(x)dx \,, \tag{2.1}$$

and

$$\lim_{t\to\infty} F(t) = \lim_{t\to\infty} H(t) = S \,. \tag{2.2}$$

Then we can define a third integral which also converges to S; namely,

$$V_1(t) = \int_a^t v(x)dx \,, \tag{2.3}$$

where

$$v(x) = (f(x) - Rh(x))/(1 - R) \,, \qquad R \neq 1 \,. \tag{2.4}$$

Since we are desirous of creating an integral which converges more rapidly to the same limit, then we might consider R as a parameter to be chosen to achieve this means. Further, suppose we adopt the intuitive notion that $V_1(t)$ should converge more rapidly to S than $F(t)$ or $H(t)$ if,

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† Texas Technological College, Lubbock, Texas. Work on this paper was performed while the authors were employed by LTV Electrosystems, Inc., Greenville, Texas.

$$\lim_{x\to\infty} \frac{v(x)}{f(x)} = \lim_{x\to\infty} \frac{v(x)}{h(x)} = 0 . \tag{2.5}$$

Then we see that (2.5) can be satisfied if,

$$R = \lim_{x\to\infty} \frac{f(x)}{h(x)} , \tag{2.6}$$

provided this limit is not one or zero.

Thus we have

$$V_1(t) = (F(t) - RH(t))/(1 - R) . \tag{2.7}$$

Finally, we note that if we are to have a method which is equally applicable to strictly numerical data, then (2.7) may not suffice, since (2.6) may not be known.

To escape this problem we approximate R in (2.7) by $f(x)/h(x)$. This leads us to the following transformation:

$$V(t) = (F(t) - R(t)H(t))/(1 - R(t)) , \tag{2.8}$$

where $R(t) = f(t)/h(t) \neq 1$.

Since (2.8) is not yet in the form we wish to consider, we will refrain for the present from giving a formal definition. The problem of using (2.8) is that it requires two integrals which converge to the same limit. To remove this difficulty, we proceed as follows. Let $g(t) \in C^{(1)}$ on $(-\infty, \infty)$ such that $\lim_{t\to\infty} g(t) = \infty$ and $g^{-1}(a)$ exists, where g^{-1} denotes the inverse of g. Then

$$\lim_{t\to\infty} \int_a^{g(t)} f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx . \tag{2.9}$$

Moreover, if in the integral on the left we let $x = g(z)$, then we have

$$\lim_{t\to\infty} \int_{g^{-1}(a)}^t f(g(z))g'(z)dz = \lim_{t\to\infty} \int_a^t f(x)dx = S . \tag{2.10}$$

We note now that (2.10) gives two improper integrals of the first kind which converge to the same limit and differ only by the lower limits, which are finite. Therefore,

$$\frac{1}{1-R}\left\{\int_a^{g^{-1}(a)} f(x)dx + \int_{g^{-1}(a)}^t [f(x) - Rf(g(x))g'(x)]dx\right\} \to S \tag{2.11}$$

as $t \to \infty$. Thus, from the reasoning above, the second integral in (2.11) will converge more rapidly if R is selected appropriately. In order to approximate R in the most natural way, it is necessary to alter (2.11), in the same manner as we altered (2.3). This, in light of (2.9), leads to the following formal definition.

Definition 1. Let $g \in C^{(1)}$ and $f \in C^{(0)}$ on $[a, \infty)$. Moreover, assume $g(t) \geqq a$ when $t \geqq a$ and that $\lim_{t\to\infty} g(t) = \infty$. Then we define the generalized G-transform by

$$G[F; g; t] = \frac{F(t) - R(t)F(g(t))}{1 - R(t)} , \tag{2.12}$$

where

$$R(t) = f(t)/f(g(t))g'(t) \tag{2.13}$$

and

$$F(t) = \int_a^t f(x)dx \rightarrow S \neq \pm \infty \quad \text{as} \quad t \rightarrow \infty . \tag{2.14}$$

We assume in (2.13) that either $f(g(t))g'(t) \neq 0$, or, if $f(g(t))g'(t) = 0$ at $t = t_0$, then

$$\lim_{t \to t_0} \frac{f(t)}{f(g(t))g'(t)} \tag{2.15}$$

exists and we define $R(t_0)$ to be that limit.

The following two definitions will be necessary for clarity in the remaining portion of this paper.

Definition 2. If $A(t)$ and $B(t)$ are two functions defined on the real numbers such that $\lim_{t\to\infty} A(t) = A \neq \pm\infty$ and $\lim_{t\to\infty} B(t) = B \neq \pm\infty$, then we say $A(t)$ converges uniformly better than $B(t)$ on (a, b) if and only if

$$|A - A(t)| < |B - B(t)| \tag{2.16}$$

for every $t \in (a, b)$.

Definition 3. If $A(t)$ and $B(t)$ are as in Definition 2, then we say $A(t)$ converges more rapidly than $B(t)$ if and only if

$$\lim_{t\to\infty} \frac{A - A(t)}{B - B(t)} = 0 . \tag{2.17}$$

THEOREM 1. *For every function g such that* $\lim_{t\to\infty} R(t) = R_g \neq 1$ *or* 0, $G[F; g; t] \rightarrow S$ *as* $t \rightarrow \infty$. *Moreover,* $G[F; g; t]$ *converges more rapidly than* $F(t)$ *or* $F(g(t))$.

Proof. By definition,

$$\frac{S - G[F; g; t]}{S - F(t)} = \frac{1}{1 - R(t)} \left\{ 1 - R(t) \left[\frac{S - F(g(t))}{S - F(t)} \right] \right\} \tag{2.18}$$

and

$$\frac{S - G[F; g; t]}{S - F(g(t))} = \frac{1}{1 - R(t)} \left\{ \frac{S - F(t)}{S - F(g(t))} - R(t) \right\} . \tag{2.19}$$

But by L'Hospital's theorem,

$$\lim_{t\to\infty} \frac{S - F(t)}{S - F(g(t))} = R_g .$$

Thus,

$$\lim_{t\to\infty} \frac{S - G[F; g; t]}{S - F(t)} = \lim_{t\to\infty} \frac{S - G[F; g; t]}{S - F(g(t))} = 0 , \tag{2.20}$$

and $G[F; g; t]$ converges more rapidly than $F(t)$ or $F(g(t))$. Moreover, since $F(t) \rightarrow S$ as $t \rightarrow \infty$, we see from (2.20) that $G[F; g; t]$ converges to S also.

THEOREM 2. *For every function g such that* $\lim_{t\to\infty} R(t) = R_g \neq 1$, $G[F; g; t] \rightarrow S$

as $t \to \infty$. Moreover, $G[F; g; t]$ converges more rapidly than $F(g(t))$.

Proof. The result follows from (2.19) in the same manner as Theorem 1.

The effectiveness of a transformation may be determined, in the limit, using the concept of more rapid convergence; however, for numerical purposes, the concept of uniformly better convergence is more useful. The following corollary to Theorem 1 is then of interest.

COROLLARY 1. *If $\lim_{t\to\infty} R(t) = R_g \neq 1$ or 0, then there exists a $T > a$ such that $G[F; g; t]$ converges uniformly better than $F(t)$ or $F(g(t))$ on (T, ∞).*

THEOREM 3. *Let $R(t) \neq 1$ on (α, β). $G[F; g; t]$ converges uniformly better than $F(t)$ on (α, β) if, and only if*

$$-2 < \frac{R(t)}{1 - R(t)} \frac{F(g(t)) - F(t)}{E(t)} < 0 \tag{2.21}$$

on (α, β), where $E(t) = S - F(t)$.

Proof. Since

$$-2 < \frac{R(t)}{1 - R(t)} \frac{F(g(t)) - F(t)}{E(t)} < 0\,, \tag{2.22}$$

then

$$-1 < \frac{S[1 - R(t)] - [F(t) - R(t)F(g(t))]}{[1 - R(t)][S - F(t)]} < 1\,. \tag{2.23}$$

Therefore,

$$\left|\frac{S - G[F; g; t]}{S - F(t)}\right| < 1 \tag{2.24}$$

for $t \in (\alpha, \beta)$, and the sufficiency is established. A reversal of the steps proves the necessity.

As was mentioned earlier, the generalized G-transform can be used to derive some more specific transformations. In particular, suppose we consider the simple G-transform of [1]. In connection with this transformation, we have the following theorem.

THEOREM 4. *A necessary and sufficient condition that $G[F; g; t]$ be exact for all $t \geqq t_0 > a$ when*

$$f(t) = C_1 \exp[-C_2 t]\,, \qquad C_2 > 0 \text{ and } t \geqq t_0\,, \tag{2.25}$$

is that $g(t) = t - K_1$, where $K_1 \neq 0$.

Further, when $g(t) = t - K_1$, the generalized G-transform reduces to the "simple" G-transform.

Proof. That $g(t) = t - K_1$ leads to the G-transform is obvious. The fact that it can be derived by imposing the condition that the "generalized-G" be exact on a particular type of function can be shown as follows.

In general, when $t > a$,

$$G[F; g; t] = S + \frac{-\int_t^\infty f(x)dx + R(t) \int_{g(t)}^\infty f(x)dx}{1 - R(t)}\,. \tag{2.26}$$

Thus for $G[F; g; t]$ to be exact at $t \geqq t_0 > a$ it is necessary and sufficient that the second term on the right in (2.26) be identically zero when $t \geqq t_0$. Now suppose $f(t)$ is given by (2.25). Without loss of generality we may assume $C_1 = 1$ in (2.25). Then a necessary and sufficient condition for exactness is that

$$(2.27)\qquad -\frac{1}{C_2}\exp\left[-C_2 t\right] + \frac{1}{C_2}\frac{\exp\left[-C_2 t\right]}{\exp\left[-C_2 g(t)\right]g'(t)}\exp\left[-C_2 g(t)\right] = 0\,,\quad t \geqq t_0\,,$$

or

$$(2.28)\qquad g'(t) = 1\,.$$

Therefore, it is necessary that

$$(2.29)\qquad g(t) = t - K_1\,,\qquad K_1 \neq 0\,.$$

Substitution of (2.29) in (2.26) shows the condition is also sufficient and the theorem follows.

The previous theorem shows that the G-transform of [1] can be obtained from (2.12) by requiring that (2.12) be exact on exponential functions. This suggests that some new transforms might be obtained by requiring (2.12) to be exact on different classes of functions. This is in fact the case, as the next theorem shows.

THEOREM 5. *A necessary and sufficient condition that $G[F;\, g;\, t]$ be exact for all $t \geqq t_0 > a$ when*

$$(2.30)\qquad f(t) = C_1 t^{-C_2}\,,\qquad C_2 > 1\,,$$

is that

$$(2.31)\qquad g(t) = t/K_2\,,\qquad K_2 \neq 1\,.$$

Proof. The result follows in exactly the same manner as Theorem 4. This suggests the following definition.

Definition 4. Let $f \in C^{(0)}$ on $[a, \infty)$. Also, suppose $f(t)$ has at most countably many zeros. Then we define the B-transform by

$$(2.32)\qquad B[F, t; k] = G[F; t/k; kt] = \frac{F(kt) - R(kt)F(t)}{1 - R(kt)}\,,$$

where

$$(2.33)\qquad R(kt) = kf(kt)/f(t)$$

and $k > 1$.

THEOREM 6. *For every k such that* $\lim_{t\to\infty} R(kt) = R(k) \neq 0, 1$, $B[F, t; k]$ *converges more rapidly to S than $F(kt)$ or $F(t)$.*

Proof. The result is a special case of Theorem 1.

Before proceeding to the next theorem, we need the following lemma.

Lemma 1. *If $f(t) = O(t^{-\alpha})$ as $t \to \infty$, $\alpha > 1$, then*

$$(2.34)\qquad \lim_{t\to\infty}\,[\ln t]\,[F(kt) - F(t)] = 0\,,\qquad k > 1\,.$$

Proof. Since $f(t) = O(t^{-\alpha})$ as $t \to \infty$ for $\alpha > 1$, then there exists $M > 0$ and $N > 0$ such that for all $t \geqq N$, $|f(t)| \leqq Mt^{-\alpha}$. Choose p sufficiently large so that $ak^p \geqq N$. Then

$$\left|p\int_{ak^p}^{ak^{p+1}} f(x)dx\right| \leqq p\int_{ak^p}^{ak^{p+1}} |f(x)|dx$$

$$(2.35)\qquad \begin{aligned} &\leqq pM\int_{ak^p}^{ak^{p+1}} x^{-\alpha}dx \\ &= \frac{pM}{1-\alpha}[(ak^{p+1})^{1-\alpha} - (ak^p)^{1-\alpha}] \\ &= \frac{Ma^{1-\alpha}}{1-\alpha}[k^{1-\alpha} - 1]p\left(\frac{1}{k^{\alpha-1}}\right)^p . \end{aligned}$$

However, $k > 1$ implies $k^{\alpha-1} > 1$, and hence

$$(2.36)\qquad \begin{aligned} \lim_{p\to\infty} p\left(\frac{1}{k^{\alpha-1}}\right)^p &= \lim_{p\to\infty}\frac{p}{(k^{\alpha-1})^p} \\ &= \lim_{p\to\infty}\frac{1}{(k^{\alpha-1})^p \ln k^{\alpha-1}} = 0 . \end{aligned}$$

Consequently,

$$(2.37)\qquad \lim_{p\to\infty} p\int_{ak^p}^{ak^{p+1}} f(x)dx = 0 .$$

Let $t = ak^p$. Then $t \to \infty$ as $p \to \infty$, and

$$(2.38)\qquad \begin{aligned} p\int_{ak^p}^{ak^{p+1}} f(x)dx &= \left[\frac{\ln t - \ln a}{\ln k}\right]\int_t^{kt} f(x)dx \\ &= \frac{1}{\ln k}\left[\ln t\int_t^{kt} f(x)dx - \ln a\int_t^{kt} f(x)dx\right]. \end{aligned}$$

Since,

$$(2.39)\qquad \lim_{t\to\infty}\int_t^{kt} f(x)dx = 0 ,$$

then

$$(2.40)\qquad \lim_{t\to\infty}[\ln t]\,[F(kt) - F(t)] = 0 ,$$

by (2.38).

THEOREM 7. *If* $B[F, t; k]$ *converges for some* $k = k_0$ *and* $f(t) = O(t^{-\alpha})$ *as* $t \to \infty$, $\alpha > 1$, *then* $\lim_{t\to\infty} B[F, t; k_0] = S$.

Proof. Assume that $B[F, t; k_0] - F(t) \to A \neq 0$ as $t \to \infty$. Thus

$$(2.41)\qquad \frac{[\ln t][F(k_0t) - F(t)]}{[\ln t][1 - R(k_0t)]} = \frac{F(k_0t) - F(t)}{1 - R(k_0t)} \to A \neq 0 \quad \text{as} \quad t\to\infty .$$

By Lemma 1, $[\ln t][F(k_0t) - F(t)] \to 0$ as $t \to \infty$, and therefore, $[\ln t][1 - R(k_0t)] \to 0$ as $t \to \infty$. Now let $t = k_0^p$, $p = 1, 2, \cdots$. Then, since $k_0 > 1$, $t \to \infty$ if and only if $p \to \infty$. Thus

$$(2.42)\qquad \lim_{t\to\infty}[\ln t][1 - R(k_0t)] = \lim_{p\to\infty}[\ln k_0]p\left[1 - \frac{a_{p+1}}{a_p}\right] = 0 ,$$

where $a_p = k_0^p f(k_0^p)$. Further, since $\lim_{t\to\infty} R(k_0 t) = 1$, then $(a_{p+1})/a_p$ is eventually positive and, by (2.42),

$$\lim_{p\to\infty} p\left(1 - \frac{a_{p+1}}{a_p}\right) = 0 . \tag{2.43}$$

Therefore, by Raabe's test, the series $\sum_{p=1}^{\infty} a_p$ diverges. However, $f(t) = O(t^{-\alpha})$ as $t \to \infty$ implies that

$$|a_p| = k_0^p |f(k_0^p)| \leqq k_0^p M (k_0^p)^{-\alpha} = M/(k_0^{\alpha-1})^p \tag{2.44}$$

and $\sum_{p=1}^{\infty} M/(k_0^{\alpha-1})^p$ converges for $\alpha > 1$, $k_0 > 1$. But then $\sum_{p=1}^{\infty} a_p$ converges, which is contradictory, and the theorem follows.

An interesting class of functions for which $B[F, t; k]$ will be useful is considered in the following theorem. For these functions, one would expect $B[F, t; k]$ to be of more value than $G[F; t - k; t + k]$, which is the simple G-transform of [1]. This is true, since the $R(t)$ associated with $G[F; t - k; t + k]$ has the limit 1, but that limit is neither 1 nor 0 in $B[F, t; k]$.

THEOREM 8. *If*

$$f(x) = \frac{\sum_{i=0}^{N} a_i x^{n-i}}{\sum_{i=0}^{M} b_i x^{m-i}} = \frac{u(x)}{v(x)} , \tag{2.45}$$

u and v relatively prime, $a_0, b_0 \neq 0$, $n - N \geqq 0$, $m - M \geqq 0$, $n + 1 < m$, and $a > x_0$, the maximum of the zeros of v, then

$$B[F, t; k] \to S = \int_a^\infty f(x)dx \quad as \quad t \to \infty . \tag{2.46}$$

Proof. This follows from Theorem 1.

In general, the problem of determining g is very difficult. Fortunately, however, the transformation which yields the integral exactly for a certain class of functions may work very well on the integral of functions which differ markedly from the exact class. This is well demonstrated in [1]. A theorem which may be of some use in determining g is the following.

THEOREM 9. *A necessary and sufficient condition that $G[F; g; t]$ be exact when $t \geqq t_0 > a$ is that $R(t)$ is constant when $t \geqq t_0$.*

Proof. From (2.26) it is necessary and sufficient that

$$\int_t^\infty f(x)dx = \frac{f(t)}{f(g(t))g'(t)} \int_{g(t)}^\infty f(x)dx , \qquad t \geqq t_0 , \tag{2.47}$$

or

$$\frac{f(g(t))g'(t)}{\int_{g(t)}^\infty f(x)dx} = \frac{f(t)}{\int_t^\infty f(x)dx} . \tag{2.48}$$

Thus, it is necessary that

$$\ln \int_{g(t)}^\infty f(x)dx = \ln C \int_t^\infty f(x)dx , \tag{2.49}$$

where C is a constant. Thus we must have

$$\int_{g(t)}^{\infty} f(x)dx = C \int_{t}^{\infty} f(x)dx . \tag{2.50}$$

This implies that

$$f(g(t))g'(t) = Cf(t) . \tag{2.51}$$

Hence, $R(t)$ must be constant when $t \geqq t_0$. Substitution shows the condition to be sufficient and the theorem follows.

3. A Limit Transformation. As in [1], if the limit of (2.33), $R(k)$ can be computed, then a "limit" transformation corresponding to $B[F, t; k]$ may be defined.

Definition 5. For $R(k) \neq 1, \;\; k > 1$, let

$$D[F, t; k] = \frac{F(kt) - R(k)F(t)}{1 - R(k)} . \tag{3.1}$$

The following theorems are then easily proved.

THEOREM 9. *$D[F, t; k]$ converges to S as $t \to \infty$.*

THEOREM 10. *$D[F, t; k]$ converges to S more rapidly than $F(t)$ as $t \to \infty$.*

THEOREM 11. *If $R(k) \neq 0, 1$, then $D[F, t; k]$ converges to S more rapidly than $F(kt)$.*

TABLE I

k	t	$F(kt)$	*Abs. Error*	$D[F, t; k]$	*Abs. Error*
1.1	10	−0.3001108580	0.10535	−0.4243361090	0.01887
1.1	20	−0.3566812162	0.04878	−0.4094517881	0.00399
1.1	40	−0.3819408831	0.02352	−0.4063907730	0.00093
1.1	80	−0.3939105581	0.01155	−0.4056925918	0.00023
1.2	10	−0.3101611965	0.09530	−0.4225255147	0.01706
1.2	20	−0.3610196178	0.04445	−0.4090969121	0.00363
1.2	40	−0.3839651752	0.02150	−0.4063115812	0.00085
1.2	80	−0.3948892711	0.01058	−0.4056738534	0.00021
4/3	6	−0.2513206547	0.15414	−0.4583000217	0.05283
4/3	12	−0.3364785082	0.06899	−0.4154304430	0.00997
4/3	24	−0.3726815576	0.03278	−0.4076673771	0.00220
4/3	48	−0.3894710391	0.01599	−0.4059886380	0.00052
4/3	96	−0.3975662010	0.00790	−0.4055969904	0.00013
2	6	−0.3101611965	0.09530	−0.4379948608	0.03253
2	12	−0.3610196178	0.04445	−0.4118780391	0.00641
2	24	−0.3839651752	0.02150	−0.4069107327	0.00145
2	48	−0.3948892711	0.01058	−0.4058133670	0.00035
2	96	−0.4002220246	0.00524	−0.4055547780	0.00009
4	6	−0.3610196178	0.04445	−0.4205836463	0.01512
4	12	−0.3839651752	0.02150	−0.4085665015	0.00310
4	24	−0.3948892711	0.01058	−0.4061791556	0.00071
4	48	−0.4002220246	0.00524	−0.4056409744	0.00018
4	96	−0.4028569999	0.00261	−0.4055129095	0.00005

TABLE II

k	t	$F(kt)$	*Abs. Error*	$B[F, t; k]$	*Abs. Error*
1.1	10	−0.3001108580	0.10535	−0.3912093754	0.01426
1.1	20	−0.3566812162	0.04878	−0.4020060773	0.00346
1.1	40	−0.3819408831	0.02352	−0.4046133578	0.00085
1.1	80	−0.3939105581	0.01155	−0.4052576711	0.00021
1.2	10	−0.3101611965	0.09530	−0.3924347312	0.01303
1.2	20	−0.3610196178	0.04445	−0.4022976964	0.00317
1.2	40	−0.3839651752	0.02150	−0.4046851170	0.00078
1.2	80	−0.3948892711	0.01058	−0.4052755065	0.00019
4/3	6	−0.2513206547	0.15414	−0.3713086935	0.03416
4/3	12	−0.3364785082	0.06899	−0.3974238614	0.00804
4/3	24	−0.3726815576	0.03278	−0.4034958738	0.00197
4/3	48	−0.3894710391	0.01599	−0.4049807169	0.00048
4/3	96	−0.3975662010	0.00790	−0.4053490378	0.00012
2	6	−0.3101611965	0.09530	−0.3832090047	0.02226
2	12	−0.3610196178	0.04445	−0.40014148003	0.00532
2	24	−0.3839651752	0.02150	−0.4041572658	0.00131
2	48	−0.3948892711	0.01058	−0.4051445448	0.00032
2	96	−0.4002220246	0.00524	−0.4053898475	0.00008
4	6	−0.3610196178	0.04445	−0.3945768170	0.01089
4	12	−0.3839651752	0.02150	−0.4028233347	0.00264
4	24	−0.3948892711	0.01058	−0.4048157377	0.00065
4	48	−0.4002220246	0.00524	−0.4053080977	0.00016
4	96	−0.4028569999	0.00261	−0.4054306283	0.00003

4. Examples. Consider the function

$$f(x) = \frac{-1}{(x-1)(x-2)} = O(x^{-2}) \quad \text{as} \quad x \to \infty \,. \tag{4.1}$$

Then

$$\int_4^\infty \frac{1}{(x-1)(x-2)}\,dx = -\ln \tfrac{3}{2} \cong -0.4054651081 \,. \tag{4.2}$$

Thus

$$R(t) = \frac{k[t^2 - 3t + 2]}{k^2t^2 - 3kt + 2} \tag{4.3}$$

and

$$\lim_{t\to\infty} R(t) = \frac{1}{k} \,. \tag{4.4}$$

The limit transformation is

$$D[F, t; k] = \frac{k}{k-1}\int_4^{kt} \frac{-1}{(x-1)(x-2)}\,dx - \frac{1}{k-1}\int_4^{t} \frac{-1}{(x-1)(x-2)}\,dx \,, \tag{4.5}$$

and the corresponding transformation is

$$B[F, t; k] = \frac{k^2t^2 - 3kt + 2}{(k^2 - k)t^2 + 2 - 2k} \int_4^{kt} \frac{-1}{(x-1)(x-2)}\,dx - \frac{k[t^2 - 3t + 2]}{(k^2 - k)t^2 + 2 - 2k} \int_4^{t} \frac{-1}{(x-1)(x-2)}\,dx\,. \tag{4.6}$$

Simpson's rule was used to approximate the integrals with $h = 0.25$. Table I above compares $D[F, t; k]$ with $F(kt)$ and Table II compares $B[F, t; k]$ with $F(kt)$.

TABLE III

k	t	$F(t + k)$	*Abs. Error*	$G[F, t; k]$	*Abs. Error*
0.5	11.5	−0.3101611965	0.09530	−0.3566133594	0.04885
0.5	23.5	−0.3610196178	0.04445	−0.3829919019	0.02247
0.5	47.5	−0.3839651752	0.02150	−0.3946597272	0.01081
0.5	95.5	−0.3948892711	0.01058	−0.4001662411	0.00530
1	5	−0.1823275323	0.22314	−0.2791364853	0.12633
1	11	−0.3101611965	0.09530	−0.3553877200	0.05008
1	23	−0.3610196178	0.04445	−0.3827362752	0.02273
1	47	−0.3839651752	0.02150	−0.3946009590	0.01086
1	95	−0.3948892711	0.01058	−0.4001521312	0.00531
2	10	−0.3101611965	0.09530	−0.3527413592	0.05272
2	22	−0.3610196178	0.04445	−0.3822071608	0.02326
2	46	−0.3839651752	0.02150	−0.3944814843	0.01098
2	94	−0.3948892711	0.01058	−0.4001236849	0.00534
4	8	−0.3101611965	0.09530	−0.3465038842	0.05896
4	20	−0.3610196178	0.04445	−0.3810713674	0.02439
4	44	−0.3839651752	0.02150	−0.3942344778	0.01123
4	92	−0.3948892711	0.01058	−0.4000658697	0.00540
8	16	−0.3610196178	0.04445	−0.3784305402	0.02703
8	40	−0.3839651752	0.02150	−0.3937055792	0.01176
8	88	−0.3948892711	0.01058	−0.3999464059	0.00552

Referring to [1] and the transformation $G[F, t; k]$ we have for $R_1(t; k)$ described in that paper

$$R_1(t; k) = \frac{f(t+k)}{f(t)} = \frac{t^2 - 3t + 2}{t^2 + (2k - 3)t + (k^2 - 3k + 2)} \tag{4.7}$$

and hence

$$\lim_{t\to\infty} R_1(t; k) = 1\,. \tag{4.8}$$

Thus,

$$G[F, t; k]_1 = \frac{t^2 + (2k-3)t + (k^2 - 3k + 2)}{2kt + k^2 - 3k} \int_4^{t+k} \frac{-1}{(x-1)(x-2)}\,dx - \frac{t^2 - 3t + 2}{2kt + k^2 - 3k} \int_4^{t} \frac{-1}{(x-1)(x-2)}\,dx\,. \tag{4.9}$$

A comparison of $G[F, t; k]$ with $F(t + k)$ is given in Table III. Note that even though $G[F, t; k]$ gives a better approximation of $F(\infty)$ than $F(t + k)$, both $D[F, t; k]$ and $B[F, t; k]$ give significantly better approximations with much less data. That is, for example, both $B[F, 20; 1.1]$ and $D[F, 20; 1.1]$ are better than $G[F, 95.5; 0.5]$, the latter being the best approximation obtainable for $F(\infty)$ through G with $t + k = 96$ and a step size of .25 in the integration. The reader should therefore note that the B and D transforms have given better approximations to $F(\infty)$, using only information from the interval (4, 22), than the G-transform gives using the interval (4, 96). Moreover, all of these transforms give better results than Simpson's rule alone.

LTV Electrosystems, Inc.
P. O. Box 1056
Greenville, Texas 75402

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