On the Relations between Eigenvalues and Eigenvectors for Matrices Resulting from Pre and Post Multiplication by the Transpose

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Abstract. Relationships between the eigenproblem associated with the matrices $A^T A$ and $AA^T$ are derived. If the larger problem is solved first, then the eigenvalues and eigenvectors associated with the smaller problem may easily be computed from the derived relationships.

Consider a nonsquare matrix $A$ of rank $m$ and of dimension $c \times d$. The product $A^T A$ is symmetric and positive semidefinite, and hence possesses an eigenvector-eigenvalue relationship of the form

$$S^T (A^T A) S = D,$$

where $S$ is a normal orthogonal matrix containing eigenvectors as columns, and $D$ is a diagonal matrix containing eigenvalues as the diagonal elements. $S$ and $D$, both of dimension $d \times d$, can be ordered so that any zero eigenvalues, resulting from $A^T A$ having excess dimension over the rank $m$, are at the bottom:

$$D = \begin{bmatrix} E & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix},$$

where $E$ is an $m \times m$ submatrix with nonzero diagonal elements. Correspondingly, the ordered $S$ can be partitioned as

$$S = [S_m ; S_p],$$

so that the left-hand part $S_m$ contains those eigenvectors associated with nonzero eigenvalues. Here $p = d - m$.

The positive semidefinite property of $A^T A$ assures that the diagonal elements of $E$ are positive, hence $E$ can be "square-rooted":

$$E = E^{1/2} E^{1/2}\text{ with } E^{1/2} \text{ being a diagonal matrix containing the square roots of the nonzero eigenvalues of } A^T A.$$\text{ with } E^{1/2} \text{ being a diagonal matrix containing the square roots of the nonzero eigenvalues of } A^T A.$$

The basic relationship (1) can be modified by premultiplication by $S$ to give

$$A^T A S = S D.$$

If (5) is written in partitioned form, the following result becomes obviously valid:

$$A^T A S_m = S_m E.$$

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A normal orthogonal vector set can be constructed as follows: First substitute (4) into (6) and premultiply by $S_m^T$, then premultiply and postmultiply by $E^{-1/2}$ to give

\begin{equation}
(AS_m E^{-1/2})^T (AS_m E^{-1/2}) = I.
\end{equation}

Equation (6) is a good starting point for investigating the eigenproblem of $AA^T$. Premultiplying (6) by $A$ and postmultiplying that result by $E^{-1/2}$ gives

\begin{equation}
(AA^T)(AS_m E^{-1/2}) = AS_m E^{1/2} = (AS_m E^{-1/2})E.
\end{equation}

Now, from (7) and (8) it follows that

\begin{equation}
(AS_m E^{-1/2})^T (AA^T)(AS_m E^{-1/2}) = E.
\end{equation}

Now denote the eigenvector-eigenvalue relationship for $AA^T$ similarly to (1) as

\begin{equation}
R^T (AA^T) R = D^* ,
\end{equation}

where $R$ and $D^*$ are of dimension $c \times c$. Reorder $R$ and $D^*$ to obtain

\[
R = [R_m; R_q] \quad \text{and} \quad D^* = \begin{bmatrix} E^* & 0 \\ 0 & 0 \end{bmatrix}.
\]

Here $q = c - m$ and $E^*$ is an $m \times m$ diagonal submatrix.

The following properties are well known:

(1) $E^* = E$.

(2) $R_q$ is arbitrary, subject to the restrictions $R_q^T R_m = 0$, $R_q^T R_q = I$.

The important result of the paper is that

\begin{equation}
R_m = AS_m E^{-1/2}
\end{equation}

which follows from an examination of (9).

In conclusion, if the larger problem (i.e., the eigenproblem associated with $A^TA$) is solved first, then the complete eigenproblem associated with $AA^T$ may be computed by using the above-mentioned well-known properties together with the result (11).

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