An Iterative Procedure for Computing the Maximal Root of a Positive Matrix

By T. L. Markham

I. Introduction. A. Kolmogoroff proved the following theorem [2].

**Theorem A.** Suppose $A$ is a positive matrix of order $n$. Then $A$ is similar to a positive generalized stochastic matrix with each row sum equal to the maximal positive characteristic root, $\lambda(A)$, of $A$.

The proof of Theorem A depends upon the fact that each positive matrix has a positive characteristic vector corresponding to the maximal positive characteristic root.

Professor A. T. Brauer developed a practical method in [1] for computing the maximal characteristic root of a positive matrix to any desired degree of accuracy. In particular, he proved

**Theorem B.** Let $A$ be a positive square matrix of order $n$, and let $R(r)$ denote the largest (smallest) row sum of $A$. If $\epsilon > 0$, there exists a matrix, $F(\epsilon)$, which is similar to $A$ such that $R^* - r^* < \epsilon$, where $R^*$ ($r^*$) denotes the largest (smallest) row sum of $F(\epsilon)$.

It is known that $r \leq \lambda(A) \leq R$ [1, Theorem 13, p. 21], and the inequality is strict unless $R = r$. Thus it follows immediately from Theorem B that $\lambda(A)$ can be computed to any desired degree of accuracy.

We shall offer a new method for computing the maximal root of a positive matrix.

II. An Iterative Procedure for Computing the Maximal Root. We shall use the following notation. Suppose $A$ is a positive matrix of order $n$. Let $R_i$ denote the $i$th row sum of $A$, $R = \max_i \{R_i\}$, and $r = \min_i \{R_i\}$.

**Theorem 1.** Suppose $A$ is a positive matrix of order $n$, with $R > r$. Let $S = \text{diag} (R_1, \ldots, R_n)$. Then $S^{-1}AS$ is a positive matrix with each row sum in the interval $(r, R)$.

**Proof.** Assume, without loss of generality, that $R = R_1 \geq R_2 \geq \cdots \geq R_n = r$. Then $S^{-1}AS = (a_{ij}R_j/R_i)$ is a positive matrix. Moreover,

$$\sum_{j=1}^{n} \frac{a_{ij}R_j}{R_i} < \sum_{j=1}^{n} \frac{a_{ij}R_1}{R_i} = R,$$

and

$$\sum_{j=1}^{n} \frac{a_{ij}R_j}{R_i} > \sum_{j=1}^{n} \frac{a_{ij}R_n}{R_i} = r,$$

for $i = 1, 2, \ldots, n$.

Hence each row sum of $S^{-1}AS$ lies in the interval $(r, R)$.

If we denote $B_1 = S^{-1}AS$, then we can transform $B_1$ by a similarity transformation, obtaining $B_2$, such that each row sum of $B_2$ lies in the interval determined

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by the largest and smallest row sums of $B_1$. We continue in this manner and obtai
a sequence $B_1, B_2, \cdots, B_n, \cdots$ such that the difference of the largest row sum
and the smallest row sum of $B_i$ is decreasing at each step. Theorem 2 assures us
that we do obtain convergence to a positive generalized stochastic matrix.

Theorem 2. Suppose $A$ is a positive matrix of order $n$, and let $B_1, B_2, \cdots,$
$B_n, \cdots$ denote the sequence of matrices obtained from $A$ by the procedure outlined
above. Assume $a_{kl}/R_k = \min_i \{a_{i1}/R_i\}$. Then

$$\max \text{ row sum of } B_n - \min \text{ row sum of } B_n \leq (1 - a_{kl}/R_k)^n (R - r).$$

Proof. Note the following inequalities:

$$\text{ith row sum of } B_1 = \sum_{j=1}^n a_{ij} \frac{R_j}{R_i} \leq \sum_{j=1}^{n-1} a_{ij} \frac{R_j}{R_i} + \frac{a_{in}r}{R_i} = \sum_{j=1}^{n-1} \frac{a_{ij}(R_j - r)}{R_i} + r
= \left(1 - \frac{a_{in}}{R_i}\right)(R - r) + r = \left(1 - \frac{a_{in}}{R_i}\right)r + \frac{a_{in}}{R_i} r.

Assume $\min_i \{a_{in}/R_i\} = a_{pm}/R_p$. Then

(1) \text{ith row sum of } B_1 \leq (1 - a_{pm}/R_p)R + (a_{pm}/R_p)r \text{ for each } i.

Similarly, we have

(2) \text{ith row sum of } B_1 \geq (1 - a_{kl}/R_k)r + (a_{kl}/R_k)R \text{ for each } i,

where $a_{kl}/R_k = \min_i \{a_{i1}/R_i\}$.

Using (1) and (2), it follows that

$$\max \text{ row sum of } B_1 - \min \text{ row sum of } B_1
\leq (1 - a_{pm}/R_p)R + (a_{pm}/R_p)r - [(1 - a_{kl}/R_k)r + (a_{kl}/R_k)R]
= (1 - a_{pn}/R_p + a_{kl}/R_k)(R - r) < (1 - a_{kl}/R_k)(R - r).

Hence the result holds when $n = 1$; i.e.

(3) \max \text{ row sum of } B_1 - \min \text{ row sum of } B_1 < (1 - a_{kl}/R_k)(R - r).

In the same manner, it is true that

(4) \max \text{ row sum of } B_2 - \min \text{ row sum of } B_2
\leq \left(1 - \min_i \left\{\frac{a_{i1}R_1}{\sum_{j=1}^n a_{ij}R_j}\right\}\right)(\max \text{ row sum of } B_1 - \min \text{ row sum of } B_1).

But

(5) \frac{a_{i1}R_1}{\sum_{j=1}^n a_{ij}R_j} \geq \frac{a_{i1}R_1}{\sum_{j=1}^n a_{ij}R_1} \geq \frac{a_{kl}}{R_k} \text{ for each } i.

Hence

$$\min_i \left\{\frac{a_{i1}R_1}{\sum_{j=1}^n a_{ij}R_j}\right\} \leq \frac{a_{kl}}{R_k}.

Finally, with (4) and (5), we have

(6) \max \text{ row sum of } B_2 - \min \text{ row sum of } B_2 \leq (1 - a_{kl}/R_k)^2(R - r).
The remainder of the proof is an easy induction.
Theorem 1 yields a simple method for the computation of the maximal positive root of a positive matrix.

III. An Example. The following computations were performed at the University of North Carolina at Charlotte Computer Center, and the author gratefully acknowledges the use of an algorithm prepared by Professors D. Nixon and L. Davis.

Let

\[
A = \begin{pmatrix}
4 & 7 & 25 & 2 \\
1 & 12 & 5 & 18 \\
3 & 6 & 15 & 2 \\
5 & 2 & 2 & 1
\end{pmatrix}.
\]

<table>
<thead>
<tr>
<th></th>
<th>min row sum</th>
<th>max row sum</th>
</tr>
</thead>
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<tr>
<td>A</td>
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<tr>
<td>B₁</td>
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<td>B₈</td>
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</table>

All numbers are truncated at four decimal places, and the computations were performed in double-precision arithmetic.

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