Error Bounds for the Gauss-Chebyshev Quadrature Formula of the Closed Type

By M. M. Chawla

1. Introduction. We are concerned with the Gauss-Chebyshev quadrature formula of the closed type,

\[ \int_{-1}^{1} (1 - t^2)^{-1/2} f(t) dt = \sum_{k=0}^{n} A_k f(t_k) + E_n(f) \quad (n \geq 2) \]

with the abscissas

\[ t_k = \cos \left( \frac{k\pi}{n} \right), \quad k = 0, \cdots, n, \]

and the Christoffel numbers

\[ A_0 = A_n = \pi/2n, \quad A_k = \pi/n, \quad k = 1, \cdots, n - 1. \]

The quadrature formula (1) is exact for all polynomials of degree \( \leq 2n - 1 \). For a general discussion of the Gauss formulas of the closed type, see Krylov [1, Chapter 9].

The usual real-variable theory estimate for the error \( E_n(f) \) is given (Krylov [1, p. 171]) in terms of derivatives of \( f \):

\[ E_n(f) = -\frac{\pi}{2^{2n-1}} \frac{f^{(2n)}(\eta)}{(2n)!} \]

for some \( \eta \in [-1, 1] \). The error expression (2) is valid for the class of functions which are \( 2n \)-times differentiable. In most cases, the exact value of \( \eta \) will be unknown, and the estimate \( \max_{-1 \leq t \leq 1} |f^{(2n)}(t)| \) is used. But in many cases it will be far from convenient to obtain \( f^{(2n)} \) or the bounds on it.

In the following, we use the complex-variable method to obtain a contour integral representation for \( E_n(f) \), applied to analytic functions, and give bounds for the error in terms of the size of the integrand in the complex plane.

2. Error Bounds. Let \( f(t) \) be analytic on \([-1, 1]\), then it can be continued analytically so as to be single-valued and analytic in a domain \( D \) of the \( z \)-plane containing the interval \([-1, 1]\) in its interior.

Let \( C \) be a closed contour in \( D \) enclosing the interval \([-1, 1]\) in its interior and let \( U_{n-1} = 2^{n-1} \prod_{k=1}^{n-1} (t - t_k) \) be the Chebyshev polynomial of the second kind. On applying the residue theorem to the contour integral

\[ \frac{1}{2\pi i} \int_{C} \frac{f(z)dz}{w(z)}, \quad w(t) = (t^2 - 1)U_{n-1}(t), \]

we get

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Multiplying (4) by the weight \((1 - t^2)^{-1/2}\) and integrating on \([-1, 1]\), there results the quadrature formula (1), with the error

\[ E_n(f) = \frac{i}{\pi} \int_0^1 \frac{Q_{n-1}(z)f(z)dz}{U_{n-1}(z)(z^2 - 1)} \]

where we have put

\[ Q_n(z) = \frac{1}{2} \int_{-1}^1 (1 - t^2)^{1/2} U_n(t) \frac{dt}{z - t} . \]

In a recent paper (Chawla [2]), the following result was proved. For sufficiently large \(|z|\),

\[ \left| \frac{Q_n(z)}{U_n(z)} \right| \leq \frac{\pi}{2^{2n+2}} |z|^{-2n-1} . \]

Taking \(C: |z| = R\) with sufficiently large \(R\), from (5) and (7), we find

\[ |E_n(f)| \leq \frac{\pi}{2^{2n-1}} \frac{R^2 M(R)}{(R^2 - 1) R^{2n}} , \]

where \(M(R) = \max_{|z| = R} |f(z)|\).

These error bounds are simple to obtain and they will not be unduly pessimistic, but are valid for the class of functions which are continuable analytically in a sufficiently large domain of the \(z\)-plane containing the range of integration \([-1, 1]\).

We obtain next estimates for \(E_n(f)\) for all functions analytic on \([-1, 1]\). For this purpose, we introduce the ellipse \(\mathcal{E}_\rho\) \((\rho > 1)\) defined by

\[ z = \tfrac{1}{2}(\xi + \xi^{-1}) , \quad \xi = \rho e^{i\theta} \quad (0 \leq \theta \leq 2\pi) \]

with foci at \(z = \pm 1\) and semiaxes \(\tfrac{1}{2}(\rho + \rho^{-1})\) and \(\tfrac{1}{2}(\rho - \rho^{-1})\).

Let \(f(t)\) be analytic on \([-1, 1]\). Then, for some \(\rho > 1\), \(f\) can be continued analytically into the closure of an ellipse \(\mathcal{E}_\rho\). It has been proved (Chawla [2]) that for \(z\) on \(\mathcal{E}_\rho\),

\[ Q_n(z) = (\pi/2)\xi^{-n-1} . \]

Since on \(\mathcal{E}_\rho\),

\[ U_n(z) = (\xi^{n+1} - \xi^{-n-1})/(\xi - \xi^{-1}) \]

and by virtue of (10), (5) becomes

\[ E_n(f) = i \int_{|\xi|=\rho} \frac{f(\tfrac{1}{2}(\xi + \xi^{-1}))d\xi}{\xi((\xi^{2n} - 1)^{1/2})} \]

or, equivalently,

\[ E_n(f) = i \int_{\mathcal{E}_\rho} \frac{f(z)dz}{(z^2 - 1)^{1/2}[(z \pm (z^2 - 1)^{1/2})^{2n} - 1]} . \]
where the sign in the integrand is chosen so that \(|z \pm (z^2 - 1)^{1/2}| > 1\). From (12) follows the following estimate for the error:

\[
|E_n(f)| \leq 2\pi M(\rho)/(\rho^{2n} - 1)
\]

where \(M(\rho) = \max |f| \text{ on } |\xi| = \rho\).

By experimenting with various "admissible" \(\rho\), a conservative upper bound can be established. A similar remark applies to the estimate (8).

3. Example. Consider the estimation of error in the evaluation of the integral

\[
J = \int_{-1}^{1} (1 - t^2)^{-1/2} \frac{at}{4 + t} = \frac{\pi}{(15)^{1/2}} = 0.811155735192
\]

by the quadrature formula (1). For \(n = 4\), the approximate value found by the quadrature formula is \(=0.811155845096\). Thus, the true error \(E_4 \approx -0.000001099\).

In this case, the real-variable estimate (2) gives \(|E_4| \leq 0.0000012\). Taking \(R = 3.5\), the estimate (8) gives \(|E_4| \leq 0.0000023\).

However, evaluating the contour integral in (13), we find

\[
E_4 = -\frac{2\pi}{(15)^{1/2}[(4 + (15)^{1/2})^8 - 1]} \approx -0.0000001099
\]

which is the exact error.

Department of Mathematics  
Indian Institute of Technology  
Hauz Khas, New Delhi-29  
India