

The Riemann Hypothesis and Pseudorandom Features of the Möbius Sequence

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Abstract. A study of the cumulative sums of the Möbius function on the Atlas Computer of the Science Research Council has revealed certain statistical properties which lead the authors to make a number of conjectures. One of these is that any conjecture of the Mertens type, viz.

$$|M(N)| = \left| \sum_{n=1}^N \mu(n) \right| < k(\sqrt{N})$$

where k is any positive constant, is false, and indeed the authors conjecture that

$$\text{Lim sup } \{M(x)(x \log \log x)^{-1/2}\} = \sqrt{(12)/\pi}.$$

The Riemann zeta function is defined for $R(s) > 1$ by the series $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, and the definition is completed by analytic continuation. Riemann's hypothesis is that the "complex zeros" all occur where $R(s) = \frac{1}{2}$. It has now been verified for the first 2,000,000 complex zeros (Rosser and Schoenfeld, [8]), but this is not a very good reason for believing that the hypothesis is true. For in the theory of the zeta function, and in the closely allied theory of the distribution of prime numbers, the iterated logarithm $\log \log x$ is often involved in asymptotic formulae, and this function increases extremely slowly. The first zero off the line $R(s) = \frac{1}{2}$, if there is one, might have an imaginary part whose iterated logarithm is, say as large as 10, and, if so, it might never be practicable to find this zero by calculation. The plausibility of this argument is increased when we recall the refutation by Littlewood [5] of the conjecture that $\pi(x)$, the number of primes less than x , is always less than the logarithmic integral, $\text{li}(x)$, a conjecture that is presumably true at least as far as $x = 10^9$ (see Ingham [4, p. 7]). It is possible that Littlewood had this kind of argument in mind when he said (Littlewood [6]), "In the spirit of this anthology (an anthology of partly baked ideas) I should also record my feeling that there is no imaginable *reason* why it (the Riemann hypothesis) should be true."

The aim of the present note is to suggest a "reason" for believing Riemann's hypothesis.

The Möbius function is defined by $\mu(n) = (-1)^k$ if the positive integer n is the product of k different primes, $\mu(1) = 1$, and $\mu(n) = 0$ if n has any repeated factor. It is known (see, for example, Titchmarsh [9, p. 315]) that a necessary and sufficient condition for the truth of the Riemann hypothesis is that $M(x) = O(x^{1/2+\epsilon})$, for all $\epsilon > 0$, where $M(x) = \sum_{n \leq x} \mu(n)$. The condition $M(x) = O(x^{1/2+\epsilon})$ would be true if the Möbius sequence $\{\mu(n)\}$ were a random sequence, taking the values $-1, 0$, and 1 , with specified probabilities, those of -1 and 1 being equal.

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More generally, if we first select a subsequence from $\{\mu(n)\}$ by striking out all the terms for which $\mu(n) = 0$, and if this subsequence were 'equiprobably random,' i.e. if the value -1 and 1 each had (conditional) probability $\frac{1}{2}$, then the condition $M(x) = O(x^{1/2+\epsilon})$ would still be true. Of course a deterministic sequence can at best be 'pseudorandom' in the usual incompletely defined sense in which the term is used, and of course all our probability arguments are put forward in a purely heuristic spirit without any claim that they are mathematical proofs.

As a matter of fact there are *a priori* reasons, that is without looking at the numerical support, for believing the following conjecture:

Conjecture A. The sums of $\mu(n)$ in blocks of length N , where N is large, have asymptotically a normal distribution with mean zero and variance $6N/\pi^2$.

A priori reasons for believing this conjecture. Note first that $\mu(n) = 0$ when n is a multiple of 4, so there are zeros in the sequence $\{\mu(n)\}$ at regular intervals of 4, and similarly at regular intervals of 9 and so on. So, if we write ν_i for the number of values of n (in a block of length N) for which $\mu(n) = i$, we would expect ν_0 to be very close to its expected value

$$N - N(1 - 2^{-2})(1 - 3^{-2})(1 - 5^{-2}) \dots = N(1 - 6\pi^{-2}).$$

The numerical evidence for this statement is given in Tables 2 and 3 in the appendix.

Now if n is large and known to be square-free it is likely to have a fair number of factors, and therefore by the theory of the roulette wheel (with two sectors instead of 37 or 38) the probabilities that the number of factors is odd or even are nearly equal. Thus the probability that $\mu(n) = 1$ (or -1) is near to $3\pi^{-2}$ and tends to this value when the range in which n is known to lie tends to infinity. Hence the expectation of $\mu(n)$ is 0.

The probability distribution of ν_1 , conditional on a knowledge of ν_0 , is binomial with mean $\frac{1}{2}(N - \nu_0)$ and variance $\frac{1}{4}(N - \nu_0)$. Allowing for the near-constancy of ν_0 , the unconditional distribution of ν_1 would be expected to be binomial with mean $3N\pi^{-2}$ and variance $3N\pi^{-2}/2$. As a matter of fact the variance does not depend on the near-constancy of ν_0 since

$$\begin{aligned} \text{var}(\nu_1 - \nu_{-1}) &= E(\nu_1 - \nu_{-1})^2 \\ &= E\{E[(\nu_1 - \nu_{-1})^2 | \nu_0]\} \\ &= E(N - \nu_0) = 6N\pi^{-2}. \end{aligned}$$

We have here assumed that, given ν_0 , ν_1 has a 'heads-and-tails' binomial distribution. Its sample size is, of course, $N - \nu_0$.

This completes our *a priori* argument for believing Conjecture A, and even if Conjecture A is only approximately true it is so much stronger than the condition $M(x) = O(x^{1/2+\epsilon})$ that we feel its approximate truth would still support that condition.

We now describe the numerical test of Conjecture A, which was performed with $N = 1000$.

We computed $M(1000r + 1000) - M(1000r)$ (where we write $M(0) = 0$) for $r = 0(1)49,999$ on the Chilton Atlas (as a "background" job) but the values $r = 34,000(1)34,999$ were lost owing to a machine fault. Column (ii) of Table 1 gives the frequencies with which

$$M(1000r + 1000) - M(1000r)$$

lies in the ranges given by column (i). The “expected values” of these frequencies, based on Conjecture A, are listed in column (iii). For example,

$$7688 = \frac{49}{\sigma\sqrt{2\pi}} \int_{0.5}^{10.5} \exp(-\frac{1}{2}x^2/\sigma^2) dx,$$

where $\sigma^2 = 6000/\pi^2$. It will be seen that the fit is extremely good, in fact $\chi^2 = 24.2$ with 22 degrees of freedom. Thus Conjecture A is not merely to be expected *a priori*, by mathematical common sense, but it is well supported by the numerical data. As we said before, we believe therefore that there is a good “reason” for believing the Riemann hypothesis, apart from the calculation of the first 2,000,000 zeros.

TABLE 1.
Frequencies of the values of $\sum \mu(n)$ ($1000r < n \leq 1000(r + 1)$),
for 49000 values of r .

(i) Range	(ii) Frequencies	(iii) Expectations
- 101 to - 110	0	1.0
- 91 to - 100	1	4.9
- 81 to - 90	16	20.6
- 71 to - 80	63	76.5
- 61 to - 70	248	240
- 51 to - 60	635	652
- 41 to - 50	1491	1465
- 31 to - 40	2797	2832
- 21 to - 30	4711	4645
- 11 to - 20	6604	6478
- 1 to - 10	7513	7688
0	793	794
1 to 10	7779	7688
11 to 20	6450	6478
21 to 30	4601	4645
31 to 40	2842	2832
41 to 50	1477	1465
51 to 60	651	652
61 to 70	216	240
71 to 80	88	76.5
81 to 90	16	20.6
91 to 100	7	4.9
101 to 110	1	1.0
	<u>49000</u>	<u>49000.0</u>

Although $\{\mu(n)\}$ is not a random sequence it is tempting to apply the law of the iterated logarithm (for example, Feller [1, p. 157]) to the subsequence obtained by deleting the values of n for which $\mu(n) = 0$. In this manner we generate a second conjecture, which, however, is less probable than Conjecture A and for which it is difficult to obtain numerical support. But it is of some interest to consider it.

Conjecture B.

$$\limsup \{M(x)(x \log \log x)^{-1/2}\} = \sqrt{(12)/\pi}.$$

Conjecture B contradicts Mertens's conjecture that $|M(x)| < x^{1/2}$, even in the extended form $|M(x)| < Cx^{1/2}$ for any constant C . When $C = \frac{1}{2}$ this modification of Mertens's conjecture was refuted numerically by Neubauer [7]: a breakdown occurred, for example at $x = 7.76 \times 10^9$. Also a conjecture of Pólya's, closely related to that of Mertens, was refuted by Haselgrove [3], who believed further that his method could be applied, with 1000 times as much calculation, to disprove Mertens's conjecture. In the light of this evidence, Mertens's conjecture is improbable, and our Conjecture B is somewhat supported by its inconsistency with it.

Appendix. Distribution of ν_0 for $N = 1,000,000$

In Table 2 we give the values of ν_0 , i.e. the number of cases of $\mu(n) = 0$, in the first, second, . . . , 33rd block of length a million. We stopped at this point owing to the machine fault previously mentioned: a single supervisor fault caused the output for Table 1 to be lost at the 35th million and for Table 2 at the 34th million.

TABLE 2

<i>Million</i>	ν_0	<i>Million</i>	ν_0
1	392,074	18	392,088
2	392,049	19	392,039
3	392,104	20	392,037
4	392,037	21	392,072
5	392,103	22	392,084
6	392,076	23	392,096
7	392,053	24	392,047
8	392,101	25	392,096
9	392,061	26	392,071
10	392,051	27	392,071
11	392,073	28	392,065
12	392,078	29	392,079
13	392,073	30	392,065
14	392,095	31	392,083
15	392,083	32	392,077
16	392,093	33	392,076
17	392,057		

Using the 33 values of ν_0 given in Table 2, the estimated standard deviation is only 19.1. (The average number of zeros in each block of a million is very nearly 392,073.) This suggests that nearly always ν_0 is nearly constant in the sense that, for large N , the standard deviation of ν_0 is $o(\sqrt{N})$, which for possible future reference we call Conjecture C. The total expected number of zeros of the sequence $\{\mu(n)\}$ in the first 33,000,000 is $33,000,000(1 - 6\pi^{-2}) = 12,938,405.6$ and the observed number is 12,938,407, an astonishingly close fit, better than we deserved.

The values of ν_0 for a few further values of N are shown in Table 3. On the basis of this table we might even strengthen Conjecture C to (Conjecture D) 'the variance of ν_0 for large N is a constant.'

TABLE 3

N	ν_0	$E(\nu_0)$	$\nu - E(\nu_0)$
25,000,000	9,801,820	9,801,822.5	- 2.5
50,000,000	19,603,656	19,603,645	+11
75,000,000	29,405,440	29,405,467	-27
100,000,000	39,207,306	39,207,290	+16

The conjecture of the near constancy of ν_0 in each block of length N must be interpreted in an average ('probabilistic') sense, whether or not it is expressed in the form of Conjecture C. It would *not* be correct to assume that ν_0 is always close to its expected value; in fact it will sometimes though very rarely happen that $\nu_0 = N$. This will happen, for example, in the block $(M + 1, M + 2, \dots, M + N)$ if simultaneously $M \equiv -1 \pmod{4}$, $M \equiv -2 \pmod{9}$, $M \equiv -3 \pmod{25}$, \dots , $M \equiv -N \pmod{p_N^2}$, where p_N is the N th prime. These congruences can be solved by Sun-Tsu's theorem (see, for example, Good [2, p. 759]). The value of M so obtained will be something like N^{2N} . Thus the Möbius sequence $\{\mu(n)\}$ contains arbitrarily long runs of zeros, but these long runs presumably occur extremely rarely.

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