On Designs of Maximal \((+1,-1)\)-Matrices of Order \(n=2\) (mod 4). \(\Pi^\ast\)

By C. H. Yang

Abstract. Finding maximal \((+1,-1)\)-matrices \(M_{2m}\) of order \(2m\) (with odd \(m\)) constructible in the standard form

\[
\begin{pmatrix}
A & B \\
-B^T & A^T
\end{pmatrix}
\]

is reduced to the finding of two polynomials \(C(w), D(w)\) (corresponding to the circulant submatrices \(A, B\)) satisfying

\[(*) \quad |C(w)|^2 + |D(w)|^2 = \frac{1}{2}(m-1),\]

where \(w\) is any primitive \(m\)th root of unity. Thus, all \(M_{2m}\) constructible by the standard form (see [4]) can be classified by the formula \((*)\). Some new matrices \(M_{2m}\) for \(m = 25, 27, 31\), were found by this method.

Let \(M_{2m}\) be a maximal \((+1,-1)\)-matrix of order \(2m\) and let \(S = ((s_i))\) be the circulant matrix of order \(m\) with the first row entries \(s_i\) \((0 \leq i \leq m - 1)\), all zero but \(s_1 = 1\).

When \(m\) is odd, it is known that (for \(m \leq 27\), except \(m = 11, 17\); see [1]—[4]), \(M_{2m}\) can be constructed by the following matrix: **

\[(1) \quad R = \begin{pmatrix} A & B \\ -B^T & A^T \end{pmatrix}, \quad \text{where} \quad A = \sum_{k=0}^{m-1} a_k S^k, \quad B = \sum_{k=0}^{m-1} b_k S^k \text{ with} \quad a_k \text{ and } b_k, 1 \text{ or } -1, \text{ and } T \text{ indicates the transposed matrix. Then the gramian matrix of } R \text{ becomes}

\[RR^T = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix},\]

where \(P\) is equal to

\[(2) \quad AA^T + BB^T = 2 \left( mI + \sum_{k=1}^{m-1} S^k \right),\]

where \(I\) is the identity matrix of order \(m\).

By applying to the both sides of \((2)\) the transformation \(L\) which transforms \(S\) into a diagonal matrix \(W = [w_1, \ldots, w_m]\) with \(w_j\), all distinct \(m\)th roots of unity, (namely, \(L(S) = U^*SU = W\), where \(U\) is unitary and \(*\) indicates the conjugate transpose; see [5]) we obtain

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** For \(n = 25\), without circulancy of submatrices \(A\) and \(B\), see [2]; also see Addition of this paper.
\[ A(w)A(w^{m-1}) + B(w)B(w^{m-1}) = 2 \left( m + \sum_{k=1}^{m-1} w^k \right) \]

where

\[ A(w) = \sum_{k=0}^{m-1} a_k w^k, \quad B(w) = \sum_{k=0}^{m-1} b_k w^k, \]

and \( w \) is any \( m \)th root of unity.

Since \( w \) and \( w^{m-1} \) are conjugate to each other, (3) is also equivalent to

\[ |A(w)|^2 + |B(w)|^2 = 2 \left( m + \sum_{k=1}^{m-1} w^k \right). \]

Let \( p \) and \( q \) be respectively the numbers of \(-1\)'s in each row of \( A \) and \( B \). By replacing 1 by 0 and \(-1\) by 1 in \( A \) and \( B \) and performing the similar process as above, we obtain the following formula corresponding to (4).

\[ |C(w)|^2 + |D(w)|^2 = p + q + r \sum_{k=1}^{m-1} w^k, \]

<table>
<thead>
<tr>
<th>Table I</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m )</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>9</td>
</tr>
<tr>
<td>13</td>
</tr>
<tr>
<td>15</td>
</tr>
<tr>
<td>19</td>
</tr>
</tbody>
</table>

or

\[ 1 + w + w^4 + w^{10} \]

or

\[ 1 + w + w^3 + w^4 \]

or

\[ 1 + w + w^2 + w^4 + w^5 + w^6 + w^9 \]

or

\[ 1 + w + w^2 + w^5 + w^9 \]

or

\[ 1 + w + w^2 + w^3 + w^4 + w^6 \]

or

\[ 1 + w + w^2 + w^3 + w^6 + w^{12} \]

or

\[ 1 + w + w^2 + w^3 + w^9 + w^{12} \]

or

\[ 1 + w + w^2 + w^3 + w^{12} \]

or

\[ 1 + w + w^2 + w^3 + w^7 + w^{12} \]

or

\[ 1 + w + w^2 + w^3 + w^{11} + w^{14} + w^{16} \]

or

\[ 1 + w + w^2 + w^7 + w^{11} + w^{14} + w^{16} \]

\[ 1 + w + w^2 + w^3 + w^7 + w^{12} \]

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### Table II

<table>
<thead>
<tr>
<th>$m$</th>
<th>$C(w)$</th>
<th>$D(w)$</th>
<th>$N = N_A = N_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>$1 + w + w^6 + w^7 + w^9 + w^{10}$</td>
<td>$1 + w + w^2 + w^3 + w^6 + w^8 + w^{10} + w^{11} + w^{14} + w^{17}$</td>
<td>6</td>
</tr>
<tr>
<td>23</td>
<td>$1 + w + w^2 + w^5 + w^7 + w^{11} + w^{14}$</td>
<td>$1 + w + w^3 + w^6 + w^8 + w^{10} + w^{11} + w^{14} + w^{18}$</td>
<td>11</td>
</tr>
<tr>
<td>27</td>
<td>$1 + w + w^2 + w^6 + w^9 + w^{18} + w^{21} + w^{22}$</td>
<td>$1 + w + w^2 + w^3 + w^9 + w^{11} + w^{13} + w^{16} + w^{19} + w^{23} + w^{24}$</td>
<td>9</td>
</tr>
</tbody>
</table>

### Table III

<table>
<thead>
<tr>
<th>$k$</th>
<th>$A$ corresponding to $C(w^k)$</th>
<th>$B$ corresponding to $D(w^k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$-+-+-+-+-+-+-+-+$</td>
<td>$-+-+-+-+-+-+-+-+$</td>
</tr>
<tr>
<td>4</td>
<td>$-+-+-+-+-+-+-+-+$</td>
<td>$-+-+-+-+-+-+-+-+$</td>
</tr>
<tr>
<td>5</td>
<td>$-+-+-+-+-+-+-+-+$</td>
<td>$-+-+-+-+-+-+-+-+$</td>
</tr>
<tr>
<td>7</td>
<td>$-+-+-+-+-+-+-+-+$</td>
<td>$-+-+-+-+-+-+-+-+$</td>
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<tr>
<td>8</td>
<td>$-+-+-+-+-+-+-+-+$</td>
<td>$-+-+-+-+-+-+-+-+$</td>
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<tr>
<td>10</td>
<td>$-+-+-+-+-+-+-+-+$</td>
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</tr>
<tr>
<td>11</td>
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<tr>
<td>13</td>
<td>$-+-+-+-+-+-+-+-+$</td>
<td>$-+-+-+-+-+-+-+-+$</td>
</tr>
</tbody>
</table>
where $C(w) = \sum_{k=0}^{n-1} c_k w^k$ with $c_k = 0$ whenever $a_k = 1$ and $c_k = 1$ whenever $a_k = -1$; $D(w)$ is similarly defined. The following relation is satisfied by $r$, when $w = 1$ is put into (5).

\[(6) \quad p^2 + q^2 = p + q + r(m - 1) .\]

Similarly from (4), we have

\[(7) \quad (m - 2p)^2 + (m - 2q)^2 = 4m - 2 .\]

When $w \neq 1$, from (5), (6), and (7), we obtain

\[(*) \quad |C(w)|^2 + |D(w)|^2 = \frac{1}{2}(m - 1) .\]

All maximal $(+1, -1)$-matrices $M_{2m}$ for odd $m$ constructible by (1) with the restriction $p \leq q < \frac{1}{2}m$, for $m \leq 19$ were listed in [4].

Thus construction of maximal matrices is reduced to finding of polynomials $C(w), D(w)$ satisfying (*). Table I is obtained according to this classification for $n = 2m \leq 38$. In this table, $N, N_A, N_B$ are respectively numbers of distinct types of matrices $M_n, A, B; w$ and $w_0$ are any primitive $m$th roots of unity.

For example, when $n = 18$, $m = 9$; primitive 9th roots of unity are $w = \exp(2\pi i/9), w^2, w^4, w^k$ for $k = 5, 7, 8$. Since $w^k$ and $w^m-k$ are symmetric with respect to $w^m = 1$ which corresponds to the main diagonal, $C(w^k)$ and $C(w^m-k)$ produce designs of the same type. Consequently we can omit the cases for $w^k$ with $k \geq 5$. $C(w) = 1 + w$ and $D(w) = 1 + w^2 + w^6$ produce the corresponding designs $---+++"++"$ and $---+++"++"$ respectively. Similarly, $C(w^2) = 1 + w^2$ and $D(w^2) = 1 + w^4 + w^6$ produce the designs $---+++"++"$ respectively. Likewise, $C(w^4) = 1 + w^4$ and $D(w^4) = 1 + w^8 + w^{12}$ produce $---+++"++"$ and $---+++"++"$. When $m = 19$, it is sufficient to consider primitive roots $w = \exp(2\pi i/19), w^k$ for $2 \leq k \leq 9$. For the case $C(w) = 1 + w + w^2 + w^4 + w^7 + w^{12}$; we have $C(w^2) = 1 + w^2 + w^4 + w^6 + w^{14} + w^8, C(w^3) = 1 + w^3 + w^6 + w^{12} + w^2 + w^{17} = w^3 C(w^2)$, and $C(w^5) = 1 + w^5 + w^{10} + w + w^{16} + w^3 = w^5 C(w^{-2})$, which produce the corresponding designs $---+++"++"$ and $---+++"++"$ respectively. All of these three designs are of the same type and their finite sequences are equal to $-1, 3, 3, 3, 3, -1, -1, 3$. Similarly it can be shown easily that $C(w^k)$ for $k = 4, 6, 9$ produces designs with the finite sequences $3, -1, -1, 3, -1, 3, 3, 3, 3; for k = 1, 7, 8, it produces those with $3, 3, 3, -1, 3, -1, 3, 3, -1$. In general, it can be shown that designs produced by $C(w), w^kC(w)$, and $w^6C(w^{-1})$, are of the same type for any integers $k$ and $h$.

Table II is obtained by applying this method of finding polynomials $C(w)$ and $D(w)$ to the previously known designs for $m = 21, 23, and 27$.

For example, when $m = 27$, with primitive roots $w^k = \exp(2\pi ki/27)$, (Table III) designs of distinct types are obtained.  

Addition. The following new designs for $M_{56}, M_{54}$, and $M_{62}$ with the corresponding $C(w)$ and $D(w)$ have been found.

When $m = 25$, we have

\[
C(w) = 1 + w + w^2 + w^6 + w^8 + w^{10} + w^{11} + w^{14} + w^{15},
\]

\[
D(w) = C(w^7) = 1 + w^2 + w^6 + w^8 + w^7 + w^{14} + w^{17} + w^{20} + w^{23} .
\]
When $m = 27$, we have

$$C(w) = 1 + w + w^2 + w^3 + w^6 + w^{10} + w^{12} + w^{15} + w^{23},$$
$$D(w) = 1 + w + w^2 + w^5 + w^7 + w^9 + w^{10} + w^{17} + w^{20} + w^{21} + w^{23}.$$ 

When $m = 31$, we have

$$C(w) = 1 + w + w^2 + w^3 + w^4 + w^6 + w^{13} + w^{19} + w^{23} + w^{26},$$
$$D(w) = 1 + w + w^2 + w^3 + w^6 + w^7 + w^{10} + w^{12} + w^{14} + w^{15} + w^{17} + w^{18} + w^{24} + w^{26} + w^{28}.$$ 

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