On Designs of Maximal (+1, -1)-Matrices of Order \( n=2 \pmod{4} \). II*

By C. H. Yang

Abstract. Finding maximal (+1, -1)-matrices \( M_{2m} \) of order \( 2m \) (with odd \( m \)) constructible in the standard form

\[
\begin{pmatrix}
  A & B \\
- & -B^T & A^T
\end{pmatrix}
\]

is reduced to the finding of two polynomials \( C(w) \), \( D(w) \) (corresponding to the circulant submatrices \( A, B \)) satisfying

\[
|C(w)|^2 + |D(w)|^2 = \frac{1}{2}(m - 1)
\]

where \( w \) is any primitive \( m \)th root of unity. Thus, all \( M_{2m} \) constructible by the standard form (see [4]) can be classified by the formula (*). Some new matrices \( M_{2m} \) for \( m = 25, 27, 31 \), were found by this method.

Let \( M_{2m} \) be a maximal (+1, -1)-matrix of order \( 2m \) and let \( S = ((s_i)) \) be the circulant matrix of order \( m \) with the first row entries \( s_i \) (\( 0 \leq i \leq m - 1 \)), all zero but \( s_1 = 1 \).

When \( m \) is odd, it is known that (for \( m \leq 27 \), except \( m = 11, 17 \); see [1]—[4]), \( M_{2m} \) can be constructed by the following matrix:**

\[
R = \begin{pmatrix}
  A & B \\
- & -B^T & A^T
\end{pmatrix}, \quad \text{where } A = \sum_{k=0}^{m-1} a_k S^k, B = \sum_{k=0}^{m-1} b_k S^k \text{ with } a_k \text{ and } b_k \text{ 1 or } -1, \text{ and } T \text{ indicates the transposed matrix. Then the gramian matrix of } R \text{ becomes}
\]

\[
RR^T = \begin{pmatrix}
P & 0 \\
0 & P
\end{pmatrix},
\]

where \( P \) is equal to

\[
AA^T + BB^T = 2 \left( mI + \sum_{k=1}^{m-1} S^k \right),
\]

where \( I \) is the identity matrix of order \( m \).

By applying to the both sides of (2) the transformation \( L \) which transforms \( S \) into a diagonal matrix \( W = [w_1, \cdots, w_m] \) with \( w_j \), all distinct \( m \)th roots of unity, (namely, \( L(S) = U^*SU = W \), where \( U \) is unitary and * indicates the conjugate transpose; see [5]) we obtain

** For \( n = 25 \), without circulancy of submatrices \( A \) and \( B \), see [2]; also see Addition of this paper.

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** For \( n = 25 \), without circulancy of submatrices \( A \) and \( B \), see [2]; also see Addition of this paper.
A(w)A(w^{m-1}) + B(w)B(w^{m-1}) = 2\left(m + \sum_{k=1}^{m-1} w^k\right)

where

A(w) = \sum_{k=0}^{m-1} a_k w^k, \quad B(w) = \sum_{k=0}^{m-1} b_k w^k,

and w is any mth root of unity.

Since w and w^{m-1} are conjugate to each other, (3) is also equivalent to

|A(w)|^2 + |B(w)|^2 = 2\left(m + \sum_{k=1}^{m-1} w^k\right).

Let p and q be respectively the numbers of \(-1\)'s in each row of A and B. By replacing 1 by 0 and \(-1\) by 1 in A and B and performing the similar process as above, we obtain the following formula corresponding to (4).

|C(w)|^2 + |D(w)|^2 = p + q + r \sum_{k=1}^{m-1} w^k,

Table I

<table>
<thead>
<tr>
<th>m</th>
<th>C(w)</th>
<th>D(w)</th>
<th>N_A</th>
<th>N_B</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1 + w + w^2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>1 + w^2 + w^4</td>
<td>1 + w + w^3 + w^6</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>1 + w</td>
<td>1 + w^2 + w^4</td>
<td>1 + w^3 + w^6 + w^8</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>13</td>
<td>1 + w + w^3 + w^9</td>
<td>1 + w + w^2 + w^4 + w^6 + w^8</td>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>15</td>
<td>1 + w + w^4 + w^6</td>
<td>1 + w^2 + w^4 + w^6 + w^{10}</td>
<td>4</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>1 + w + w^4 + w^{10}</td>
<td>1 + w + w^2 + w^4 + w^6 + w^8</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>19</td>
<td>1 + w + w^2 + w^6 + w^{10} + w^{14}</td>
<td>1 + w^3 + w^4 + w^7 + w^12</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>1 + w + w^2 + w^6 + w^{12} + w^{14}</td>
<td>1 + w + w^3 + w^4 + w^8 + w^{10} + w^{14}</td>
<td>6</td>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>1 + w + w^2 + w^6 + w^{12} + w^{14} + w^{15}</td>
<td>1 + w + w^3 + w^4 + w^8 + w^{10} + w^{16}</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>1 + w + w^2 + w^3 + w^6 + w^{12}</td>
<td>1 + w + w^5 + w^7 + w^9 + w^{12} + w^{15}</td>
<td>9</td>
<td>18</td>
<td>18</td>
</tr>
</tbody>
</table>

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### Table II

<table>
<thead>
<tr>
<th>m</th>
<th>$C(w^k)$</th>
<th>$D(w^k)$</th>
<th>$N = N_A = N_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>$1 + w + w^2 + w^3 + w^4 + w^5 + w^7 + w^9 + w^{10} + w^{11} + w^{13} + w^{14} + w^{15}$</td>
<td>$1 + w + w^2 + w^3 + w^4 + w^5 + w^6 + w^7 + w^8 + w^9 + w^{10} + w^{11} + w^{14} + w^{15}$</td>
<td>6</td>
</tr>
<tr>
<td>23</td>
<td>$1 + w + w^2 + w^3 + w^4 + w^5 + w^6 + w^7 + w^8 + w^9 + w^{10} + w^{11} + w^{13} + w^{14}$</td>
<td>$1 + w + w^2 + w^3 + w^4 + w^5 + w^6 + w^7 + w^8 + w^9 + w^{10} + w^{11} + w^{14} + w^{15}$</td>
<td>9</td>
</tr>
<tr>
<td>27</td>
<td>$1 + w + w^2 + w^3 + w^4 + w^5 + w^6 + w^7 + w^8 + w^9 + w^{10} + w^{11} + w^{13} + w^{14}$</td>
<td>$1 + w + w^2 + w^3 + w^4 + w^5 + w^6 + w^7 + w^8 + w^9 + w^{10} + w^{11} + w^{14} + w^{15}$</td>
<td>11</td>
</tr>
</tbody>
</table>

### Table III

<table>
<thead>
<tr>
<th>$k$</th>
<th>$B$ corresponding to $D(w^k)$</th>
<th>$A$ corresponding to $C(w^k)$</th>
</tr>
</thead>
</table>
| 2   | $+$ + $+$ + $+$ + $+$ + $+$ + $+$ + $+$ + $+$ | $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+$ $+

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where \( C(w) = \sum_{k=0}^{n-1} c_k w^k \) with \( c_k = 0 \) whenever \( a_k = 1 \) and \( c_k = 1 \) whenever \( a_k = -1 \); \( D(w) \) is similarly defined. The following relation is satisfied by \( r \), when \( w = 1 \) is put into (5).

\[
p^2 + q^2 = p + q + r(m - 1) .
\]

Similarly from (4), we have

\[
(m - 2p)^2 + (m - 2q)^2 = 4m - 2 .
\]

When \( w \neq 1 \), from (5), (6), and (7), we obtain

\[
(*) \quad |C(w)|^2 + |D(w)|^2 = \frac{1}{2}(m - 1) .
\]

All maximal \((+1, -1)\)-matrices \( M_{2m} \) for odd \( m \) constructible by (1) with the restriction \( p \leq q < \frac{1}{2}m \), for \( m \leq 19 \) were listed in [4].

Thus construction of maximal matrices is reduced to finding of polynomials \( C(w) \), \( D(w) \) satisfying \((*)\). Table I is obtained according to this classification for \( n = 2m \leq 38 \). In this table, \( N, N_A, N_B \) are respectively numbers of distinct types of matrices \( M_n, A, B \); \( w \) and \( w_0 \) are any primitive \( m \)th roots of unity.

For example, when \( n = 18, m = 9 \); primitive 9th roots of unity are \( w = \exp(2\pi i/9), w^2, w^4, w^8 \) for \( k = 5, 7, 8 \). Since \( w^k \) and \( w^{m-k} \) are symmetric with respect to \( w^m = 1 \) which corresponds to the main diagonal, \( C(w^k) \) and \( C(w^{m-k}) \) produce designs of the same type. Consequently we can omit the cases for \( w^k \) with \( k \leq 5 \). \( C(w) = 1 + w \) and \( D(w) = 1 + w^2 + w^4 \) produce the corresponding designs \( -+++ +++++ \) and \( -+ -+ +++++ +++++ \) respectively. Similarly, \( C(w^2) = 1 + w^2 \) and \( D(w^2) = 1 + w^4 + w \) produce the designs \( -+++ +++++ +++++ \) and \( -+++ +++++ +++++ \) respectively. Likewise, \( C(w^4) = 1 + w^4 \) and \( D(w^4) = 1 + w^8 + w^2 \) produce \( -+++ +++++ +++++ \) and \( -+++ +++++ +++++ \). When \( m = 19 \), it is sufficient to consider primitive roots \( w = \exp(2\pi i/19), w^k \) for \( 2 \leq k \leq 9 \). For the case \( C(w) = 1 + w + w^2 + w^4 + w^7 + w^{12} \); we have \( C(w^3) = 1 + w^2 + w^4 + w^8 + w^{14} + w^5, C(w^3) = 1 + w^3 + w^6 + w^{12} + w^2 + w^{17} = w^3 C(w^2) \), and \( C(w^5) = 1 + w^5 + w^{10} + w + w^{16} + w^3 = w^5 C(w^{-2}) \), which produce the corresponding designs \( -+++ -+++ +++ +++++ +++++ +++++ +++++ \), \( --+ -+ +++++ +++++ +++++ +++++ +++++ \), and \( -+++ -++++ +++++ +++++ +++++ +++++ \). All of these three designs are of the same type and their finite sequences are equal to \(-1, 3, 3, 3, 3, 3, -1, -1, 3\). Similarly it can be shown easily that \( C(w^k) \) for \( k = 4, 6, 9 \) produces designs with the finite sequences \( 3, -1, 3, -1, 3, 3, 3, 3, 3; \) for \( k = 1, 7, 8 \), it produces those with \( 3, 3, 3, -1, 3, -1, 3, 3, -1 \). In general, it can be shown that designs produced by \( C(w), w^k C(w), \) and \( w^h C(w^{-1}) \), are of the same type for any integers \( k \) and \( h \).

Table II is obtained by applying this method of finding polynomials \( C(w) \) and \( D(w) \) to the previously known designs for \( m = 21, 23, \) and \( 27 \).

For example, when \( m = 27 \), with primitive roots \( w^k = \exp(2\pi ki/27) \), (Table III) designs of distinct types are obtained.

Addition. The following new designs for \( M_{56}, M_{54}, \) and \( M_{62} \) with the corresponding \( C(w) \) and \( D(w) \) have been found.

When \( m = 25 \), we have

\[
C(w) = 1 + w + w^2 + w^6 + w^8 + w^{10} + w^{11} + w^{14} + w^{15},
\]
\[
D(w) = C(w^7) = 1 + w^2 + w^6 + w^8 + w^{14} + w^{17} + w^{20} + w^{23} .
\]
When $m = 27$, we have

$$C(w) = 1 + w + w^2 + w^3 + w^6 + w^{10} + w^{12} + w^{15} + w^{23},$$
$$D(w) = 1 + w + w^2 + w^5 + w^7 + w^9 + w^{10} + w^{17} + w^{20} + w^{21} + w^{23}.$$  

When $m = 31$, we have

$$C(w) = 1 + w + w^2 + w^3 + w^4 + w^i + w^{13} + w^{19} + w^{23} + w^{26},$$
$$D(w) = 1 + w + w^2 + w^3 + w^6 + w^{10} + w^{12} + w^{14} + w^{15} + w^{17} + w^{18} + w^{24} + w^{26} + w^{28}.$$  

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