On a Theorem of Piatetsky-Shapiro and Approximation of Multiple Integrals

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Abstract. Let \( f \) be a function of \( s \) real variables which is of period 1 in each variable, and let the integral \( I \) of \( f \) over the unit cube in \( s \)-space be approximated by

\[
Q(f) = \frac{1}{N} \sum_{r=1}^{N} f(rx)
\]

(where \( x = x(N) \) is a point in \( s \)-space). For certain classes of \( f \)'s, defined by conditions on their Fourier coefficients, it is shown using methods of N. M. Korobov, that \( x \)'s can be found for which error bounds of the form \( |I(f) - Q(f)| < K(f)N^{-p} \) will be true. However, for the class of all \( f \)'s with absolutely convergent Fourier series, it is shown that there are no \( x \)'s for which a bound of the form \( |I(f) - Q(f)| = O(F(N)) \) will hold, for any \( F(N) \) which approaches zero as \( N \) goes to infinity.

In his book *Number-Theoretic Methods of Approximate Analysis*, N. M. Korobov quotes the following result of I. I. Piatetsky-Shapiro [1]:

**Theorem.** Let \( A_s \) denote the class of all functions of \( s \) real variables that have period 1 in each variable and have an absolutely convergent Fourier series:

\[
f(x) = \sum_{m_1, \ldots, m_s = -\infty}^{\infty} c(m) \exp(x \cdot m)
\]

(boldface letters denote \( s \)-tuples of real numbers; \( \exp a = e^{2\pi ia} \)). Then for any \( f \in A_s \) and any positive integer \( N \) there is a \( \theta \) such that

\[
\left| \int_0^1 \cdots \int_0^1 f(x)dx - \frac{1}{N} \sum_{r=1}^{N} f(r\theta) \right| < C \frac{\log N}{N}
\]

where \( C = C(f) \).

Though Korobov takes up this theorem in connection with methods of approximate evaluation of multiple integrals, the theorem itself does not provide such a method, as \( \theta \) depends on \( f \). The question then arises whether a \( \theta(N) \) exists which will make (2) true for all \( f \in A_s \). We answer this in the negative; but we do show that there are \( \theta \)'s which allow a stronger statement than (2) for some considerable subsets of \( A_s \).

We will denote the unit cube in \( s \)-dimensional Euclidean space by \( G_s \).

**Theorem 1.** If \( N_1, N_2, \ldots \) is an increasing sequence of positive integers, \( \theta^{(1)}, \theta^{(2)}, \ldots \) a sequence of \( s \)-tuples of real numbers, and \( F(n) \) any positive decreasing function such that \( F(n) \to 0 \) as \( n \to \infty \), then there is an \( f \in A_s \) such that

\[
\left| \int_{G_s} f - \frac{1}{N} \sum_{r=1}^{N} f(r\theta^{(i)}) \right| / F(N_i)
\]

is unbounded as \( i \to \infty \).

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Proof. $A_*$ is a Banach space, if, with the expansion (1), we define

$$
\|f\| = \sum_{m_1, \ldots, m_s = -\infty}^{\infty} |c(m)|.
$$

Define the linear functional $L_i$, $i = 1, 2, \ldots$, by

$$
L_i(f) = \frac{1}{F(N_i)} \left( \int_{a_i} f - \frac{1}{N_i} \sum_{r=1}^{N_i} f(r^{(i)}) \right).
$$

If the theorem does not hold, then $|L_i(f)| \leq C(f)$, $i = 1, 2, \ldots$ for every $f \in A_*$, where $C(f)$ is some real number. Then by the Banach-Steinhaus Theorem (see, e.g., [2]) there is a constant $K$ such that

$$
|L_i(f)| \leq K \|f\|
$$

for all $i$ and all $f \in A_*$. But if we choose $m^{(i)}$, for each $i$, in such a manner that

$$
m^{(i)} \cdot \theta^{(i)}
$$

is within $1/2N_i^2$ of an integer (and the components of $m^{(i)}$ are not all zero), and set $f_i(x) = \exp(x - m^{(i)})$, then

$$
|L_i(f_i)| = \frac{1}{N_i F(N_i)} \left| \sum_{r=1}^{N_i} \exp(r^{(i)} \cdot m^{(i)}) \right| \geq \frac{1}{2F(N_i)},
$$

contradicting (4).

If

$$
D = \sum_{m_1, \ldots, m_s = -\infty}^{\infty} d(m)
$$

is a convergent (s-tuple) series of positive constants, we shall denote by $A_*(D)$ the subset of $A_*$ consisting of those functions having expansions (1) satisfying

$$
|c(m)| \leq Cd(m), \quad -\infty < m_1, \ldots, m_s < \infty
$$

for some number $C$.

Theorem 2. If $D$ is any convergent s-tuple series of positive numbers, and $N$ is a prime number, then there are integers $a_1, a_2, \ldots, a_s$ between 0 and $N - 1$ such that

$$
\left| \int_{a_i} f - \frac{1}{N} \sum_{r=1}^{N} f(r \cdot a_i) \right| < \frac{K(f)}{N}
$$

for all $f \in A_*(D)$.

Proof. Using the expansion (1), we see that

$$
\int_{a_i} f = c(0, \ldots, 0) while
$$

$$
\frac{1}{N} \sum_{r=1}^{N} f\left(\frac{r}{N} a\right) = \sum_{m_1, \ldots, m_s = -\infty}^{\infty} c(m) \left( \frac{1}{N} \sum_{r=1}^{N} \exp\left(\frac{r}{N} a \cdot m\right) \right)
$$

$$
= c(0, \ldots, 0) + \sum_{m_1, \ldots, m_s = -\infty}^{\infty} c(m) \delta_N(a \cdot m)
$$

where $\delta_N(n)$ is 1 if $N$ divides $n$ and is 0 otherwise, and the prime on the sum indicates that the term with $m_1 = m_2 = \cdots = m_s = 0$ is omitted. Therefore

$$
\left| \int_{a_i} f - \frac{1}{N} \sum_{r=1}^{N} f\left(\frac{r}{N} a\right) \right| \leq \sum_{m_1, \ldots, m_s = -\infty}^{\infty} |c(m)| \delta_N(a \cdot m)
$$

$$
\leq C \sum_{m_1, \ldots, m_s = -\infty}^{\infty} d(m) \delta_N(a \cdot m).
$$
Let us now look at the average, for given \( N \) and \( \mathbf{m} \), of \( \delta_N(\mathbf{a} \cdot \mathbf{m}) \) over all \( s \)-tuples \( \mathbf{a} \) of integers from 0 to \( N - 1 \). Choosing \( a_j \) such that \( m_j \neq 0 \), we see that for any choice of \( a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_s \) there is just one value of \( a_j \) making \( \delta = 1 \)—since \( N \) is a prime—and \( N - 1 \) values for which \( \delta = 0 \). Thus for each \( \mathbf{m} \),

\[
\text{av} \delta_N(\mathbf{a} \cdot \mathbf{m}) = 1/N.
\]

It follows that

\[
\min_{0 \leq a_1, \ldots, a_s \leq N-1} \left| \int \mathcal{G}_s \ f - \frac{1}{N} \sum_{r=1}^{N} f\left( \frac{r}{N} \right) \mathbf{a} \right| \\
\leq \text{av} C \sum_{m_1, \ldots, m_s = -\infty}^{\infty} d(\mathbf{m}) \delta_N(\mathbf{a} \cdot \mathbf{m}) \\
\leq \frac{C}{N} \sum_{m_1, \ldots, m_s = -\infty}^{\infty} d(\mathbf{m}),
\]

proving the theorem.

This result has consequences for the numerical integration of functions satisfying certain stronger conditions. If, following Korobov, we set

\[
\overline{m} = \max (|m|, 1), \quad ||\mathbf{m}|| = \overline{m}_1 \cdot \overline{m}_2 \cdot \cdots \cdot \overline{m}_s,
\]

and denote by \( E_s^\alpha \), for \( \alpha > 1 \), the set of functions having an expansion (1) that satisfies \( |c(\mathbf{m})| \leq C(f)||\mathbf{m}||^{-\alpha} \), we have

**Corollary 1.** For each prime number \( P \) and for any positive number \( \epsilon \) there are integers \( a_1, a_2, \ldots, a_s \) such that for any \( f \in E_s^\alpha \)

\[
\left| \int \mathcal{G}_s \ f - \frac{1}{P} \sum_{r=1}^{P} f\left( \frac{r}{P} \right) \mathbf{a} \right| < K(f) P^{-\alpha - \epsilon}.
\]

**Proof.** Set \( \beta = \max (\alpha - \epsilon, 1) \) and set

\[
g(x) = \sum_{m_1, \ldots, m_s = -\infty}^{\infty} ||\mathbf{m}||^{-\alpha/\beta} \exp (\mathbf{m} \cdot \mathbf{x});
\]

and let \( \mathbf{a} \) be the \( s \)-tuple of Theorem 2. Since \( \sum t^\beta \leq (\sum t)^\beta \) whenever \( \beta \geq 1 \) and the t's are nonnegative, the quantity

\[
\sum_{m_1, \ldots, m_s = -\infty}^{\infty} |c(\mathbf{m})| \delta_P(\mathbf{a} \cdot \mathbf{m})
\]

for \( f \) is no greater than \( C(f) \) times the \( \beta \)th power of the same quantity for \( g \); and the latter is less than or equal to \( K(g)/P \).

Korobov obtains a sharper result than this; where we have \( P^{-\alpha + \epsilon} \) in (9) he has \( P^{-\alpha} \log^\beta P \) for certain values of \( \beta \). If we further restrict the class of functions we obtain a result that does not follow directly from Korobov's theorems:

Let \( L_s^\alpha \), for \( \alpha > 1 \), be the class of all functions having an expansion (1) that satisfies

\[
|c(\mathbf{m})| \leq C(||\mathbf{m}|| \log^{1+\epsilon} ||\mathbf{m}||)^{-\alpha}
\]

for some \( C = C(f) \) and \( \epsilon = \epsilon(f) > 0 \). Then we have, by a proof similar to that of the above corollary,
Corollary 2. For each prime number \( P \) there is a set of integers \( a_1, \ldots, a_s \) such that for any \( f \in L_s^a \)

\[
\left| \int_{G_s} f - \frac{1}{P} \sum_{r=1}^{P} f\left(\frac{r}{P} \cdot a \right) \right| < \frac{K(f)}{P^\alpha}.
\]

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1. N. M. Korobov, Number-Theoretic Methods of Approximate Analysis, Fizmatgiz, Moscow, 1963, p. 85. (Russian) MR 28 #716.