Computer Investigation of Landau’s Theorem

By P. S. Chiang

Abstract. Let \( f(z) = a_0 + a_1 z + \cdots \) be regular for \( |z| < 1 \) and never take the values 0 and 1; then \( |a_1| \) has a bound depending only on \( a_0 \). J. A. Jenkins gave an explicit bound (Canad. J. Math. 8 (1956), 423–425) \( |a_1| \leq 2|a_0| \{ \log |a_0| + 5.94 \} \). The author investigates the shapes for the curves \( |a_1| \leq L(a_0) \) for given \( a_0 \) by the aid of a computer and shows that although Jenkins’ result is about right when \( a_0 \) is negative, 4.38 will be the best possible constant in his form and that a much smaller estimate should be available when \( a_0 \) is positive or complex.

1. Introduction. The theorem of Landau in question may be stated in the form that if the function \( f(z) = a_0 + a_1 z + \cdots \) is regular for \( |z| < 1 \) and never takes the values 0 and 1, then \( |a_1| \) has a bound depending only on \( a_0 \). Hayman [1] gave the explicit bound \( |a_1| \leq 2|a_0| \{ \log |a_0| + 5\pi \} \) and Jenkins [2] improved it to \( |a_1| \leq 2|a_0| \{ \log |a_0| + 5.94 \} \). For a given value of \( a_0 \), there is a certain possible region of values of \( a_1 \). This region is probably not a circle \( |a_1| \leq K(a_0) \). This region will probably have a different shape when \( a_0 \) is near 0, 1 and \( \infty \). In this paper, I shall show that although Jenkins’ result is about right when \( a_0 \) is negative, 4.38 will be the best possible constant in his form and that a much smaller estimate should be available when \( a_0 \) is positive or complex.

2. Preliminaries. Let \( \lambda(\tau) \) be an elliptic modular function,
\[
\lambda(\tau) = \frac{\theta_2^4(0)/\theta_3^4(0)}{16q(1 + q^2 + q^6 + q^{12} + \cdots)^4/(1 + 2q + 2q^4 + 2q^9 + \cdots)^4}
\]
where \( q = e^{i\pi \tau} \). By a transformation
\[
\xi = (\tau - \tau_0)/(\tau - \tau_0), \quad I_m(\tau_0) > 0
\]
we have \( g(\xi) = \lambda(\tau) \) which is regular and \( g(\xi) \neq 0, g(\xi) \neq 1 \) for \( |\xi| < 1 \). Hence
\[
a_0 = g(0) = \lambda(\tau_0)
\]
and
\[
a_1 = g'(0) = \lambda'(\tau_0)2I_m(\tau_0).
\]
Thus, the problem of finding a better inequality in Landau’s theorem may be solved by tabulating \( |g'(0)| \) and \( g(0) \). Hence, the matter simply depends on calculating the elliptic modular function \( \lambda(\tau) \).

3. A Bound of \( |a_1| \) for small \( |a_0| \). When \( I_m(\tau) \) is large and hence \( |q| \) small we have \( g(0) \simeq 16q_0 \) where \( q_0 = e^{i\pi \tau_0} \) and
\[
g'(0) \simeq 16i\pi e^{i\pi \tau_0}2I_m(\tau_0).
\]
Hence

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Therefore

\[ |g'(0)| \approx \pi |g(0)| (2/\pi) \log |g(0)/16|. \]

Thus, for small \( |a_0| \), Landau’s inequality is approximately

\[ |a_1| \leq 2|a_0| \left( \log \frac{1}{|a_0|} + 2.7726 \right). \]


4.1. The Case of \( a_0 \) Real. When \( a_0 \) is real, we need only to compute the value of \( a_1 = X'(r_0) 2/m(r_0) \) against \( a_0 = \lambda(r_0) \) which varies from 1/2 to 0 and \(-1\) to 0. The other values can be obtained by the following transformations:

\[ U = 1 - W, \quad V = 1/U; \]

here of course, \( |U'| = |W'| \) and \( |V'| = |U'|/|U|^2 \). A simple computer program will give us a sufficient amount of information about the values of \( a_0 \) and \( a_1 \). I shall list only a few of them below and show the part of the curve in the attached figure. The computation is made by taking

\[ \lambda(r) = 16q(1 + q^2 + q^6 + q^{12} + q^{20})^4/(1 + 2q + 2q^4 + 2q^9 + 2q^{18})^4. \]

| \( a_0 \) | \( |a_1| \) | \( a_0 \) | \( |a_1| \) | \( a_0 \) | \( |a_1| \) | \( a_0 \) | \( |a_1| \) |
|---|---|---|---|---|---|---|---|
| 0.5 | 2.1884 | -0.1 | 1.0744 | -0.6 | 5.3195 | -1.1 | 9.6336 |
| 0.4 | 2.1177 | -0.2 | 1.9587 | -0.7 | 6.1661 | -1.2 | 10.5220 |
| 0.3 | 1.9020 | -0.3 | 2.8063 | -0.8 | 7.0203 | -1.3 | 11.4187 |
| 0.2 | 1.5263 | -0.4 | 3.6432 | -0.9 | 7.8829 | -1.4 | 12.3237 |
| 0.1 | 0.9527 | -0.5 | 4.4793 | -1.0 | 8.7538 | -1.5 | 13.2371 |

4.2. \( |a_1| \leq 2|a_0| \left( \log |a_0| + 4.38 \right) \). From the above table, we notice that the constant \( \Gamma^4(1/4)/4\pi^2 = 4.376 \cdots \) in Littlewood’s result [3] at \( a_0 = -1 \) is very sharp and by using the inequality in 3 and the numerical tabulation of \( 2|a_0| \left( \log |a_0| + 4.38 \right) \), we can read that 4.38 will be the best possible constant in Jenkins’ form.

Remark 1. In fact, we have \( |a_1| = 2.18843961 \) and \( |a_1| = 8.75375837 \) for \( a_0 = \lambda(i) = 0.50000000 \) and \( a_0 = \lambda(1 + i) = -0.99999999 \) respectively. Hence, even if we consider a few more terms in \( \lambda(r) \), almost no change in the value of \( |a_1| \) can be expected.

4.3. The Case of \( a_0 \) Complex. I shall illustrate the best possible numerical bound of \( |a_1| \) for each given \( a_0 \) with the argument \( \alpha = n\pi/10, n = 1, 2, \cdots, 10 \) in the figure. These curves are drawn from the values prepared by a computer by taking

\[ \lambda(r) = 16q(1 + q^2 + q^6 + q^{12})^4/(1 + 2q + 2q^4 + 2q^9 + 2q^{18})^4. \]

Remark 2. From the table and the figure and from the transformations \( U = 1 - W \) and \( V = 1/U \), we can obtain the values of \( a_0 \) and its corresponding values of \( |a_1| \) which suggest the shape of a possible region of values of \( a_1 \) for a given
For instance, we may draw the contour lines of $L(a_0) = \text{constant}$ in the $a_0$-complex plane. It is interesting to mention that Jenkins' result would just give concentric circles in that representation.

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