On the Numerical Solution of a Differential-Difference Equation Arising in Analytic Number Theory

By J. van de Lune and E. Wattel

1. Abstract and Introduction. In the January 1962 issue of this journal R. Bellman and B. Kotkin published a short paper under the same title as this one (cf. [1]). In that paper Bellman and Kotkin presented some of their results concerning the numerical computation of the continuous function $y(x)$, defined by

$$
y(x) = 1 \quad (0 \leq x \leq 1)$$

$$
y'(x) = -\frac{1}{x}y(x - 1) \quad (x > 1)
$$

Tables of $y(x)$ were given for $x = 1(0.0625)6$ and $x = 6(1)20$. In the process of extending these tables beyond $x = 20$ we discovered that the second table was rather inaccurate for all values of $x > 9$. Bellman and Kotkin found, for example, that $y(20) = 0.149 \cdot 10^{-8}$, whereas the actual value of $y(20)$ can be shown to be smaller than $10^{-20}$. Moreover, in view of the method used by Bellman and Kotkin, one may expect that it would be quite time consuming to compute $y(x)$ for values of $x$ up to say $x = 1,000$.

In this paper we describe a different method which enables us to compute $y(x)$ for values of $x$ up to about "as far as one would like." $lacksquare$

2. The Main Formula and Some of its Consequences. The function $y(x)$ defined in the introduction satisfies the following fundamental

**Lemma 1.**

$$
x \cdot y(x) = \int_{x-1}^{x} y(t) dt \quad (x \geq 1)
$$

*Proof.* Cf. de Bruijn [2].

A simple consequence of this lemma is

**Lemma 2.**

$$
y(x) > 0 \quad (x \geq 0)
$$

As an easy consequence of this lemma and the definition of $y(x)$ we find that $y(x)$ is monotonically decreasing on $x \geq 1$.

**Lemma 3.** $y(x)$ is concave on $x \geq 1$.

*Proof.* From the definition of $y(x)$ it follows that

$$
y(x) = 1 - \log x \quad (1 \leq x \leq 2),
$$

so that

$$
y(x) \text{ is concave on } 1 \leq x \leq 2.
$$
Also from the definition of \( y(x) \) it is easily seen that \( y(x) \) is twice differentiable on \( x > 2 \), whereas \( y(x) \) is precisely once differentiable at \( x = 2 \). On \( x > 2 \) we have

\[
y''(x) = \frac{d}{dx} \left( -\frac{1}{x} \cdot y(x - 1) \right) = \frac{1}{x^2} \cdot y(x - 1) + \frac{-1}{x} \cdot \frac{-1}{x - 1} \cdot y(x - 2) > 0.
\]

Since \( y(x) \) is concave on the intervals \( 1 \leq x \leq 2 \) and \( x > 2 \) and differentiable at \( x = 2 \), we may conclude that \( y(x) \) is concave on \( x \geq 1 \).

**Lemma 4.**

\[
y(x) < \frac{1}{2x - 1} \cdot y(x - 1), \quad (x \geq 2).
\]

**Proof.** On \( x \geq 2 \) we have by Lemma 3 that

\[
x \cdot y(x) = \int_{x-1}^{x} y(t) \, dt < \frac{1}{2} \{ y(x - 1) + y(x) \}
\]

and consequently

\[
y(x) < \frac{1}{2x - 1} \cdot y(x - 1).
\]

From Lemma 4 one easily deduces by induction that

\[
y(n) < \frac{1}{3 \cdot 5 \cdot 7 \ldots (2n - 1)} = \frac{2^n \cdot n!}{(2n)!}, \quad (n = 2, 3, 4, \ldots).
\]

Hence, for example,

\[
y(20) < \frac{2^{20} \cdot 20!}{40!} = \frac{2^{20}}{21 \cdot 22 \cdot 23 \ldots 40} < \frac{2^{20}}{20^{20}} = 10^{-20}.
\]

This rough upper bound for \( y(20) \) shows that the value of \( y(20) \) given by Bellman and Kotkin is not even of the proper order.

**3. The Numerical Computation of** \( y(x) \). Our starting point is the result of Lemma 1

\[
y(x) = 1 \quad (0 \leq x \leq 1),
\]

\[(x + 1) \cdot y(x + 1) = \int_{x}^{x+1} y(t) \, dt \quad (x \geq 0).
\]

We have already mentioned that

\[
y(x) = 1 - \log x, \quad (1 \leq x \leq 2)
\]

so that we only have to compute \( y(x) \) on \( x > 2 \).

If we approximate the integral

\[
I = \int_{x_0}^{x_0+1} y(t) \, dt, \quad (x_0 \geq 1)
\]

by means of the trapezoidal formula.
\[ \frac{1}{2n} \left\{ y(x_0) + 2 \sum_{k=1}^{n-1} y\left(x_0 + \frac{k}{n}\right) + y(x_0 + 1) \right\} \]

we obtain, because of the concavity of \( y(x) \) on \( x \geq 1 \), that

\[ (x_0 + 1)y(x_0 + 1) = \int_{x_0}^{x_0+1} y(t)dt < \frac{1}{2n} \left\{ y(x_0) + 2 \sum_{k=1}^{n-1} y\left(x_0 + \frac{k}{n}\right) + y(x_0 + 1) \right\} . \]

It follows that

\[ y(x_0 + 1) < \frac{1}{2n(x_0 + 1) - 1} \left\{ y(x_0) + 2 \sum_{k=1}^{n-1} y\left(x_0 + \frac{k}{n}\right) \right\} . \]

Thus, if one has upper bounds for \( y(x) \) at the points

\[ x_0 + k/n , \quad (k = 0, 1, 2, \ldots, n - 1) , \]

one may compute an upper bound for \( y(x_0 + 1) \).

Continuing in this way one may compute upper bounds for \( y(x) \) at the points

\[ x_0 + 1 + v/n , \quad (v = 1, 2, 3, \ldots) . \]

On the other hand, approximating \( I \) by

\[ \frac{1}{n} \sum_{k=1}^{n} y\left(x_0 + \frac{2k - 1}{2n}\right) \]

one finds, also because of the concavity of \( y(x) \) on \( x \geq 1 \), that

\[ y(x_0 + 1) > \frac{1}{n(x_0 + 1)} \sum_{k=1}^{n} y\left(x_0 + \frac{2k - 1}{2n}\right) . \]

Hence, as soon as one has lower bounds for \( y(x) \) at the points \( x_0 + (2k - 1)/2n \), \( (k = 1, 2, 3, \ldots, n) \) one may compute a lower bound for \( y(x_0 + 1) \). If one also knows lower bounds for \( y(x) \) at the points \( x_0 + k/n \), \( (k = 1, 2, 3, \ldots, n - 1) \), one can apply the same method to compute a lower bound for \( y(x_0 + 1 + 1/2n) \). Repeating this process one finds lower bounds for \( y(x) \) at the points \( x_0 + 1 + k/2n \), \( (k = 2, 3, 4, \ldots) \). As a starting point for the computations one may take of course \( x_0 = 1 \).

If one chooses the grid sizes in the above integral-approximating procedures small enough, one may expect that the corresponding upper and lower bounds for \( y(x) \) will not differ very much. Actual computations show that this is indeed the case.

Performing the computations on the Electrologica X8 of the Mathematical Centre in Amsterdam, using an ALGOL-60 program (with grid size 0.005), we found that the corresponding upper and lower bounds for \( y(x) \) were equal up to at least the first significant digit for all \( x < 100 \).

Using more refined integral-approximating formulae and smaller grid sizes we were able to compute \( y(x) \) for values of \( x \) up to at least \( x = 1,000 \). Below we include a table for \( y(x) \) with an accuracy of five or more significant figures.

Finally we will compare some of the results of Table 1 with the known asymptotic formula of de Bruijn (cf. [2])

\[ y(x) \sim \frac{e^x}{(2\pi x)^{1/2}} \exp \left\{ 1 - e^{-x} + \int_0^x e^s - \frac{1}{s} \, ds \right\} \quad (x \to \infty) , \]
where $\xi$ is the positive root of $e^t - 1 = x\xi$ and $\gamma$ is Euler's constant. De Bruijn's formula can be rewritten in terms of the exponential integral $\text{Ei}(t) = \int_{-\infty}^{t} (e^{s}/s)ds$ as

$$
y(x) \sim \frac{1}{(2\pi x)^{1/2}} \frac{1}{\xi} \cdot \exp \{-x \cdot \xi + \text{Ei}(\xi)\}, \quad (x \to \infty)
$$

which is somewhat more convenient for numerical computations. Writing $B(x)$ for de Bruijn's asymptotic approximation we have

**Table 1.** $y(x) = a(x) \cdot 10^{-b(x)}$

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<th>$b(x)$</th>
<th>$x$</th>
<th>$a(x)$</th>
<th>$b(x)$</th>
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**Table 2**

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Acknowledgement. The authors wish to thank Professor A. van Wijngaarden for his helpful suggestions concerning the numerical computations.

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Amsterdam, The Netherlands


For an extensive list of literature concerning the function \( y(x) \) we refer to

3. J. van de Lune & E. Wattel, *On the Frequency of Natural Numbers \( m \) whose Prime Divisors are all Smaller than \( m^a \)*, Mathematical Centre, Amsterdam, Report ZW 1968-007, 1968.