

Rational Chebyshev Approximations for the Error Function*

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Abstract. This note presents nearly-best rational approximations for the functions $\operatorname{erf}(x)$ and $\operatorname{erfc}(x)$, with maximal relative errors ranging down to between 6×10^{-19} and 3×10^{-20} .

In [1] Hart, et al., present rational approximations for the function

$$\operatorname{erfc}(x) \equiv 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt$$

valid for $0 \leq x \leq \alpha$, where $\alpha = 4, 8, 10$, or 20 . They carefully point out [1, p. 138] that these approximations are not useful for computing the error function

$$\operatorname{erf}(x) \equiv 1 - \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

for small x because of subtraction error, but they do not provide any alternative. Hastings' [2] approximations for $\operatorname{erf}(x)$ are no better, since they explicitly use the constant 1 as an additive term and are chosen to nearly minimize the maximum absolute error rather than the relative error. Clenshaw's [3] Chebyshev series expansions for $\operatorname{erf}(x)/x$ come close to minimizing relative error, but his approximations are somewhat inefficient because of his choice of interval and his restriction to polynomials.

For a computer subroutine with entries for both $\operatorname{erf}(x)$ and $\operatorname{erfc}(x)$, cancellation error can be avoided by evaluating $\operatorname{erf}(x)$ directly and $\operatorname{erfc}(x)$ indirectly (as $1 - \operatorname{erf}(x)$) when $\operatorname{erf}(x)$ is smaller in magnitude than $\operatorname{erfc}(x)$, and $\operatorname{erf}(x)$ indirectly and $\operatorname{erfc}(x)$ directly, otherwise. The changeover point occurs for $|x| \simeq .47$.

In this note we present nearly-best rational approximations for the functions $\operatorname{erf}(x)$ and $\operatorname{erfc}(x)$ with maximal relative errors ranging down to between 6×10^{-19} and 3×10^{-20} . The approximation forms and intervals used are

$$\begin{aligned} \operatorname{erf}(x) &\simeq xR_{lm}(x^2), & |x| &\leq .5, \\ \operatorname{erfc}(x) &\simeq e^{-x^2}R_{lm}(x), & .46875 &\leq x \leq 4.0, \\ \operatorname{erfc}(x) &\simeq \frac{e^{-x^2}}{x} \left\{ \frac{1}{\sqrt{\pi}} + \frac{1}{x^2} R_{lm}(1/x^2) \right\}, & x &\geq 4, \end{aligned}$$

where the $R_{lm}(z)$ are rational functions of degree l in the numerator and m in the denominator. The relations $\operatorname{erf}(-x) = -\operatorname{erf}(x)$ and $\operatorname{erfc}(-x) = 2 - \operatorname{erfc}(x)$ can be used to evaluate the functions for negative arguments.

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Table I. $\sum_{\ell m} = -100 \log_{10} \max \left| \frac{f(x) - f_{\ell m}(x)}{f(x)} \right|$

$f(x) = \operatorname{erf}(x), \quad |x| \leq .5$

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*****
m \ l  0      1      2      3      4      5      6      7      8
*****
0      139    313    496    688    887    1092
1      417    556*   753    960    1172   1390
2      558    702    986*  1212   1438   1666
3      800    956    1307   1465*  1698   1935
4      962    1108   1466   1626   1950*
5     1158    1338   1751   1932
*****
    
```

$f(x) = \operatorname{erfc}(x), \quad .46875 < x < 4.0$

```

*****
0      61     109    161    214    270
1     164    222*   280    340    401    462
2      376    441    506    572    638
3      440    597    666    736    806
4      502    666    824    897*
5                      1056   1132*
6                      1292   1371
7                      1532   1613*
8                      1775   1859*
*****
    
```

$f(x) = \operatorname{erfc}(x), \quad x \geq 4.0$

```

*****
0      628    756    876
1     688*   828    958   1081   1198
2     855    998*  1131   1256
3     992   1151   1287*  1415
4    1116   1283   1431   1561*
5    1232   1405   1558   1824*
*****
    
```

*Coefficients for these approximations only are given in Tables II-IV.

Table II. $\text{erf}(x) \approx f_{nn}(x) = x \sum_{j=0}^n p_j x^{2j} / \sum_{j=0}^n q_j x^{2j}, \quad |x| \leq .5$

n	j	p_j	q_j
1	0	3.67678 77	3.25845 93
	1	-9.79704 65	1.00000 00
2	0	2.13853 32237 8	1.89522 57241 5
	1	1.72227 57703 9	7.84374 57083 0
	2	3.16652 89065 8	1.00000 00000 0
3	0	2.42667 95523 05317 5	2.15058 87586 98612 0
	1	2.19792 61618 29415 2	9.11649 05404 51490 1
	2	6.99638 34886 19135 5	1.50827 97630 40778 7
	3	-3.56098 43701 81538 5	1.00000 00000 0
4	0	3.20937 75891 38469 47256 2	2.84423 68334 39170 62227 3
	1	3.77485 23768 53020 20813 7	1.28261 65260 77372 27564 5
	2	1.13864 15415 10501 55649 5	2.44024 63793 44441 73305 6
	3	3.16112 37438 70565 59694 7	2.36012 90952 34412 09349 9
	4	1.85777 70618 46031 52673 0	1.00000 00000 00000 0

Table III. $\operatorname{erfc}(x) \approx f_{nn}(x) = e^{-x^2} \sum_{j=0}^n p_j x^j / \sum_{j=0}^n q_j x^j$, $.46875 \leq x \leq 4.0$

n	j	p_j	q_j
1	0	7.3033	6.6211
	1	-2.3877	1.0000
4	0	7.37388	7.37396
	1	6.86501	1.51849
	2	3.03179	1.27955
	3	5.63169	5.35421
5	0	4.31877	1.00000
	1	92851	2.28989
	2	46956	5.19335
	3	98596	5.02732
	4	01070	2.62887
5	0	659	891
	1	075	552
	2	926	803
	3	898	761
	4	381	618
5	0	688	0000
	1	51959	0000
	2	688	0000
	3	0000	0000
	4	0000	0000

7	0	3.00459	26102	01616	005	(02)	3.00459	26095	69832	933	(02)
	1	4.51918	95371	18729	422	(02)	7.90950	92532	78980	272	(02)
	2	3.39320	81673	43436	870	(02)	9.31354	09485	06096	211	(02)
	3	1.52989	28504	69404	039	(02)	6.38980	26446	56311	665	(02)
	4	4.31622	27222	05673	530	(01)	2.77585	44474	39876	434	(02)
	5	7.21175	82508	83093	659	(00)	7.70001	52935	22947	295	(01)
	6	5.64195	51747	89739	711	(-01)	1.27827	27319	62942	351	(01)
	7	-1.36864	85738	27167	067	(-07)	1.00000	00000	00000	000	(00)
	8	1.23033	93547	97997	25272	(03)	1.23033	93548	03749	42043	(03)
	1	2.05107	83778	26071	46532	(03)	3.43936	76741	43721	63696	(03)
	2	1.71204	76126	34070	58314	(03)	4.36261	90901	43247	15820	(03)
	3	8.81952	22124	17690	90411	(02)	3.29079	92357	33459	62678	(03)
	4	2.98635	13819	74001	31132	(02)	1.62138	95745	66690	18874	(03)
	5	6.61191	90637	14162	94775	(01)	5.37181	10186	20098	57509	(02)
	6	8.88314	97943	88375	94118	(00)	1.17693	95089	13124	99305	(02)
	7	5.64188	49698	86700	89180	(-01)	1.57449	26110	70983	47253	(01)
	8	2.15311	53547	44038	46343	(-08)	1.00000	00000	00000	00000	(00)

$$\text{Table IV. } \operatorname{erfc}(x) \approx f_{mn}(x) = \frac{e^{-x^2}}{x} \left\{ \frac{1}{\sqrt{\pi}} + \frac{1}{x^2} \sum_{j=0}^n p_j x^{-2j} / \sum_{j=0}^n q_j x^{-2j} \right\}, \quad x \geq 4.0$$

n	j	p _j	q _j
1	0	-1.24368 544	4.40917 061
	1	-9.68210 364	1.00000 000
2	0	-4.25799 64355 3	1.50942 07054 5
	1	-1.96068 57372 6	9.21452 41169 4
	2	-5.16882 26218 5	1.00000 00000 0
3	0	-1.21308 27638 9978	4.30026 64345 2770
	1	-1.19903 95526 8146	4.89552 44196 1437
	2	-2.43911 02948 8626	1.43771 22793 7118
	3	-3.24319 51927 7746	1.00000 00000 0000
4	0	-2.99610 70770 35421 74	1.06209 23052 84679 18
	1	-4.94730 91062 32507 34	1.91308 92610 78298 41
	2	-2.26956 59353 96869 30	1.05167 51070 67932 07
	3	-2.78661 30860 96477 88	1.98733 20181 71352 56
	4	-2.23192 45973 41846 86	1.00000 00000 00000 00
5	0	-6.58749 16152 98378 03157	2.33520 49762 68691 85443
	1	-1.60837 85148 74227 66278	6.05183 41312 44131 91178
	2	-1.25781 72611 12292 46204	5.27905 10295 14284 12248
	3	-3.60344 89994 98044 39429	1.87295 28499 23460 47209
	4	-3.05326 63496 12323 44035	2.56852 01922 89822 42072
	5	-1.63153 87137 30209 78498	1.00000 00000 00000 00000

Table I presents the initial segments of the L_∞ Walsh arrays while Tables II, III, and IV present selected approximations. All approximations were generated using a standard version of the Remes algorithm [4] on a CDC 3600. The master function routines used continued-fraction expansions described in [1] and were verified to be accurate to at least 22S. Finally, the accuracy of the approximations as presented here was verified by comparison against the master routines using 5000 pseudo-random arguments.

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1. J. F. HART, et al., *Computer Approximations*, Siam Series in Appl. Math., Wiley, New York, 1968.
2. C. HASTINGS, JR., *Approximations for Digital Computers*, Princeton Univ. Press, Princeton, N. J., 1955. MR 16, 963.
3. C. W. CLENSHAW, *Chebyshev Series for Mathematical Functions*, National Physical Lab. Math. Tables, vol. 5, H.M.S.O., London, 1962. MR 26 #362.
4. W. J. CODY, W. FRASER & J. F. HART, "Rational Chebyshev approximations using linear equations," *Numer. Math.*, v. 12, 1968, pp. 242-251.