Chebyshev Polynomial Expansion of Bose-Einstein Functions of Orders 1 to 10*

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Abstract. Chebyshev series approximations are given for the complete Bose-Einstein functions of orders 1 to 10. This paper also gives an exhaustive presentation of the relation of this function to other functions, with the emphasis that some Fermi-Dirac functions and polylogarithms are readily computable from the given approximations. The coefficients are given in 21 significant figures and the maximal relative error for function representation ranges from 2×10^{-20} to 3×10^{-19}. These expansions are fast convergent; for example, typically six terms gives an accuracy of 10^{-8}.

1. Introduction. The Bose-Einstein function occurs in a wide variety of physical problems, in many different forms. It has been used in problems of statistical physics, quantum electrodynamics, polymer structure and electrical networks [1], [2]. We shall define the most general Bose-Einstein function by its integral representation, as follows:

$$B_p(\eta, u) = \frac{1}{\Gamma(p+1)} \int_0^u \frac{x^p dx}{e^{x-\eta} - 1}, \tag{1}$$

where η and u may be complex. For $u > \eta$ the integral is to be interpreted as a principal value. In this paper we shall only investigate the complete Bose-Einstein function for η, defined as

$$B_p(\eta) \equiv \lim_{u \to \infty} B_p(\eta, u) . \tag{2}$$

The important mathematical properties of this function are discussed in Section 2. Its relations to other functions are presented in Section 3, emphasizing the fact that some functions are readily computable from $B_p(\eta)$. Our method of obtaining the Chebyshev expansions, including discussions of actual computations and accuracy are given in Section 4. The coefficients of the Chebyshev expansion are presented in the microfiche appendix of this issue.

2. Mathematical Properties. The following properties can be found in Truesdell [3], [4], and Dingle [5]:

$$B_p(\eta) = \frac{\partial}{\partial \eta} B_{p+1}(\eta) , \tag{3}$$

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$$(4)\quad B_p(\eta) = \sum_{k=1}^{\infty} \frac{e^{k\eta}}{k^{p+1}}, \qquad \eta < 0,$$

$$(5)\quad = \sum_{k=0}^{\infty} \frac{\eta^k}{k!} \zeta(p+1-k) - \frac{\pi(-\eta)^p}{p! \sin \pi p}, \qquad 0 < -\eta < 2\pi,\ p \neq \text{integer},$$

$$(6)\quad = \sum_{k=0}^{\infty} \frac{\eta^k}{k!} \zeta(p+1-k) - \frac{\pi\eta^p}{p! \tan \pi p}, \qquad 0 < \eta < 2\pi,\ p \neq \text{integer},$$

$$(7)\quad = \sum_{k=0;\ k\neq p}^{\infty} \frac{\eta^k}{k!} \zeta(p+1-k) - \frac{\eta^p}{p!} \{\ln|\eta| - \Psi(p+1) + \Psi(1)\}, \qquad |\eta| < 2\pi,\ p = \text{integer},$$

where $\Psi(p) = (1/\Gamma(p))\,(d/dp)\Gamma(p)$, and $\zeta(p)$ is Riemann's zeta function.

$$(8)\quad \begin{aligned} B_p(\eta) = \cos \pi p B_p(-\eta) &+ 2 \sum_{k=0}^{[(p+1)/2]} \frac{\zeta(2k)\eta^{p+1-2k}}{(p+1-2k)!} \\ &+ \frac{2 \sin \pi p}{\pi} \sum_{k=[(p+3)/2]}^{\infty} \frac{\zeta(2k)(2k-p-2)!}{\eta^{2k-2p-1}}, \end{aligned}$$

where $[a]$ denotes the largest integer contained in a. This function can be treated from different routes of approach. Dingle started from the integral representation of $B_p(\eta)$ and derived useful properties, mainly from a Mellin transform. Truesdell [3] defined the function by the series (4), with Eq. (8) providing the analytic continuation. Truesdell [4] also considered a differential-difference equation of the type (3) and derived mathematical properties subject to certain boundary conditions, e.g., in this case

$$(9)\quad B_p(0) = \zeta(p+1), \qquad p > 0,$$

$$(10)\quad B_{-1}(\eta) = e^{\eta}[1 - e^{\eta}]^{-1}, \qquad \eta \neq 0,$$

or

$$(11)\quad B_0(\eta) = \ln(|1 - e^{\eta}|), \qquad \eta \neq 0.$$

He suggests this latter approach as a basis for a unified theory for most of the special functions of mathematical physics. We note parenthetically that the functions of negative integer orders are expressible in terms of elementary functions by the use of Eqs. (3) and (10).

3. Relation to Other Functions.

(i) To the hypergeometric function ${}_nF_k(a_1 \cdots a_n; b_1 \cdots b_k; z)$: It is evident that for p = integer the series (3) can be expressed in the form

$$(12)\quad B_n(\eta) = \lim_{\epsilon \to 0} [{}_{n+1}F_n(\epsilon, \cdots, \epsilon; 1, 1, \cdots, 1; e^{\eta}) - 1]\epsilon^{-(n+1)}.$$

(ii) To the polylogarithm: Using Lewin's [1] notation, we define the polylogarithm as

$$
\begin{aligned}
Li_n(z) &= \int_0^z \frac{Li_{n-1}(z)}{z}\,dz\,, \\
Li_2(z) &= \int_0^z \frac{\ln(|1-z|)}{z}\,dz\,, \qquad (13)\\
Li_1(z) &= \ln(|1-z|)\,.
\end{aligned}
$$

Then

$$Li_n(z) = B_{n-1}(\ln z)\,. \qquad (14)$$

$Li_2(z)$ has been investigated by many mathematicians, including Euler, Abel, Kummer, and Ramanujan, and is usually labeled as the dilogarithm or Euler's dilogarithm [1]. Recently Kölbig [6] presented an algorithm to compute this function to 14 significant digits.

(iii) To the Fermi-Dirac function: Define the complete Fermi-Dirac function as

$$F_p(\eta) = \frac{1}{\Gamma(p+1)} \int_0^\infty \frac{x^p dx}{e^{x-\eta}+1}\,. \qquad (15)$$

Dingle [5] has shown the following relations to be true for real η:

$$B_p(\eta) = -\text{Real part of } F_p(\eta + i\pi)\,, \qquad (16)$$

$$B_p(\eta) = \sum_{k=0}^{\infty} (2^{-p})^k F_p(2^k\eta) \qquad (17)$$

and

$$F_p(\eta) = B_p(\eta) - 2^{-p}B_p(2\eta)\,. \qquad (18)$$

The last expression suggests that the computation of the Fermi-Dirac function is facilitated readily by the Bose-Einstein function.

(iv) To the Debye function: Define the Debye function as (see [7])

$$\Gamma(p+1)D_p(x) = \int_x^\infty \frac{t^p dt}{e^t - 1}\,; \qquad (19)$$

we see the relation

$$D_p(x) = [\zeta(p+1) - B_p(0, x)]\,. \qquad (20)$$

(v) To the various zeta functions: If we start with the series representation of Riemann's zeta function,

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s}\,, \qquad (21)$$

we can generalize the function as follows:

$$\zeta(s, \alpha) = \sum_{k=0}^{\infty} (\alpha + k)^{-s}\,, \qquad (22)$$

$$F(z, s) = \sum_{k=1}^{\infty} k^{-s} z^k\,, \qquad |z| < 1\,, \qquad (23)$$

$$\Phi(z, s, \alpha) = \sum_{k=0}^{\infty} (\alpha + k)^{-s} z^k , \qquad |z| < 1 . \tag{24}$$

The last three functions are known as the generalized zeta function, Jonquière's function and Lerch's transcendent, respectively [8]. The last two functions can be appropriately continued analytically beyond the indicated circle of convergence [4]. Comparing Eq. (23) to Eq. (4), we have immediately

$$F(z, s) = B_{s-1}(\ln z) . \tag{25}$$

Notice that Jonquière's function is just a generalization of the polylogarithm.

(vi) To the exponential integral: Dingle [4] has given the following identity:

$$B_p(\eta) = \sum_{k=0}^{\infty} Ei_{p+1}(2\pi k i - \eta) , \tag{26}$$

where

$$Ei_p(z) = \int_1^{\infty} t^{-p} e^{-zt} dt \qquad (p \geqq 1, R(z) > 0) .$$

4. Chebyshev Polynomial Expansions: Approximating Forms and Computations. The advantages of expanding functions in Chebyshev polynomials are well known. Clenshaw [9] presents exhaustive discussions of comparison among Chebyshev series, best-fit polynomials and economized power series, and also methods for computing Chebyshev coefficients. In this paper, we present three sets of expansions as follows:

$$B_p(\eta) \approx e^{\eta} \sum_{k=0} a_k^{(p)} T_k^*(e^{\eta+1}) \quad \text{for } -\infty \leqq \eta \leqq -1, p = 1, 2, \cdots, 10 . \tag{27}$$

$$B_p(\eta) \approx \sum_{k=0} b_k^{(p)} T_k(\eta) - \frac{\eta^p}{p!} \ln |\eta| \quad \text{for } -1 \leqq \eta \leqq 1, p = 1, 2, \cdots, 10 . \tag{28}$$

$$B_p(\eta) \approx Q_p(\eta) + \frac{1}{2} \eta^{p+2} \sum_{k=0} c_k^{(p)} T_{2k}\left(\frac{\eta}{2}\right) - \frac{\eta^p}{p!} \ln |\eta| \qquad \text{for } -2 \leqq \eta \leqq 2, p = 1, 2, \cdots, 5 . \tag{29}$$

For the range $(1, \infty)$, one can use expansion (27) and Eq. (8) which is simple for p = integer. In the last three equations, T_n and T_n^* are the usual Chebyshev and shifted Chebyshev polynomials, Q_p is a $(p + 1)$th degree polynomial, the coefficients of which will also be given. The expansion (27) is computed from a straightforward economization of the series (4), and the expansion (28) is obtained from the use of the orthogonal property of summation, both methods being described in [9]. The expansion (29) is computed by economizing part of the series in Eq. (7), leaving out a polynomial

$$Q_p(\eta) = \sum_{k=0}^{p-1} \frac{\eta^k}{k!} \zeta(p + 1 - k) - \frac{\eta^p}{p!} [\Psi(1) - \Psi(p + 1)] - \frac{1}{2} \frac{\eta^{p+1}}{(p + 1)!} . \tag{30}$$

For the lower order functions this breaking up is advantageous because the series

$$\sum_{k=p+2}^{\infty} \frac{\eta^k}{k!} \zeta(p + 1 - k)$$

is even, due to $\zeta(-2k) = 0$, and the polynomial Q_p is a low order one. The evenness of the last series also accounts for the fact that

$$b_{p+2k+1}^{(p)} = 0\,, \qquad k = 1, 2, \cdots .$$

The above three Chebyshev expansions are rapidly convergent. For example, we need typically six terms for an accuracy of 10^{-8} and thirteen for 10^{-16}.

All computations were performed on the IBM 7094 Mod II using a package of subroutines in 70-bit (about 21 decimal digits) arithmetic, written by Dr. C. L. Lawson and associates of the Jet Propulsion Laboratory. In Tables I to X in the microfiche appendix, we present the coefficients for the expansions (27) to (29). Each expansion, with its rounded coefficients is checked by its corresponding program of function generation for 1000 pseudo-random arguments. The maximal relative error ranges from 2×10^{-20} to 3×10^{-19}. In addition, the expansions (27) and (29) were checked against each other in the cross region $-2 \leqq \eta - 1$, and the expansions (28) and (29) in $-1 \leqq \eta \leqq +1$. To further insure against gross errors, we have also used each expansion to compute the polylogarithms by Eq. (14) and spot-check against a 10-decimal table of such functions [10].

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