Reducing a Matrix to Hessenberg Form

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Abstract. It has been an open problem whether the reduction of a matrix to Hessenberg (almost triangular) form by Gaussian similarity transformations is numerically stable [2, p. 364]. We settle this question by exhibiting a class of matrices for which this process is unstable.

As a major step towards the numerical solution of the non-Hermitian algebraic eigenvalue problem, a matrix is usually first reduced to Hessenberg (almost triangular) form either by a sequence of Householder similarity transformations, [2, p. 347] or else by some form of Gaussian elimination [2, p. 353]. In practice, the latter process is favored because it requires fewer arithmetic operations. However, it is known that Householder’s reduction is numerically stable [2, p. 350], while it has been an open problem whether Gaussian elimination is stable [2, p. 364]. We settle this question by exhibiting a class of matrices for which Gaussian elimination is unstable.

The column-by-column reduction of an n by n matrix $A = (a_{ij})$ to upper Hessenberg form by Gaussian elimination produces a sequence of similarity transforms

$$A_k = (G_k^{-1}P_k^{-1})A_{k-1}(P_kG_k), \quad k = 1, 2, \ldots, n - 2,$$

of $A_0 = A$. $G_k^{-1} = (g^{(k)}_{ij})$ differs from the identity matrix only in the elements $g^{(k)}_{i,i+1}, i = k + 2, k + 3, \ldots, n$, which are chosen such that $A_k$ has zeros in positions $(k + 2, k), (k + 3, k), \ldots, (n, k)$. The permutation matrices $P_k$ are chosen such that

$$|g^{(k)}_{i,i+1}| \leq 1, \quad (i = k + 2, k + 3, \ldots, n).$$

Wilkinson [2, p. 364] points to the danger of potential worsening of the condition of the eigenvalues caused by large elements in the matrices

$$F_k^{-1} = (f^{(k)}_{ij}) = (G_k^{-1}P_k^{-1})(G_{k-1}^{-1}P_{k-1}^{-1}) \cdots (G_1^{-1}P_1^{-1}).$$

It is implicit in [2] that the largest element of $F_k^{-1}$ is bounded in magnitude by $2^{k-1}$; the theorem below shows this bound to be sharp and the example illustrates the consequences of this fact. The proof of the theorem uses a construction analogous to that of [2, p. 212].

Theorem. There exist matrices for which

$$\max_{i,j} |f^{(k)}_{ij}| = 2^{k-1}.$$

Proof. Let

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A = \begin{bmatrix}
  a_1 & a_2 & \cdots & a_{n-1} & a_n \\
  1 & & & & \\
  -1 & 1 & 0 & & \\
  & & \ddots & \ddots & \\
  & & & 0 & \\
  & & & \ddots & \\
  & & & & 0 \\
  & & & & 1
\end{bmatrix}

where \(a_1, a_2, \ldots, a_n\) are arbitrary. Then, for \(P_k = I, k = 1, 2, \ldots, n - 2, \bar{y}_{k+2,k+1} = \bar{y}_{k+1,k+1} = \cdots = \bar{y}_{n,n+1} = 1\) and \(\max |\gamma_{ij}^{(k)}| = 2^{k-1}\).

While the following example does not quite achieve the bound \(2^{k-1}\), it illustrates the effect of large elements of \(F_k\).

Example. Consider the 6 by 6 matrix

\[
B = \begin{bmatrix}
  0 & 1 & 1 & 1 & 1 & 1 \\
  1 & 0 & 0 & 0 & 0 & -1 \\
 -1 & 1 & 0 & 0 & 0 & -1 \\
 -1 & 0 & 1 & 0 & 0 & -1 \\
 -1 & 0 & 0 & 1 & 0 & -1 \\
 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0
\end{bmatrix}
\]

whose eigenvalues \(\lambda(B)\) are given to five figures in the table. (The calculations were performed on a GE645 computer by programs given in [1].) We also give approximate values of the condition numbers

\[
s_i(B) = \frac{|y_i''|}{|y_i' x_i|},
\]

where \(y_i''\) and \(x_i\) are row- and column-eigenvectors of \(B\). Since no \(s_i\) is large, all eigenvalues of \(B\) are insensitive to small perturbations in the elements of \(B\). After column-reducing \(B\) to upper Hessenberg form we obtain

\[
H = B_4 = \begin{bmatrix}
  0 & -2 & -1 & 0 & 0 & 1 \\
  1 & 0 & 0 & 0 & 1 & -1 \\
  0 & 1 & 0 & 0 & 0 & -2 \\
  0 & 0 & 1 & 0 & 4 & -4 \\
  0 & 0 & 0 & 1 & 8 & -8 \\
  0 & 0 & 0 & 0 & 8\frac{1}{2} & -8
\end{bmatrix}
\]

Since \(H\) is similar to \(B\), its eigenvalues \(\lambda(H)\) agree with those of \(B\). However, the transformed condition numbers
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\[ s_i(H) = \frac{\|y_i^H F A_i\|}{\|F A_i^H x_i\|} \]

indicate considerably greater sensitivity of the eigenvalues to perturbations in the elements of \( H \).

**Table. Numerical Results for the Example.**

<table>
<thead>
<tr>
<th>( \lambda_i(B) = \lambda_i(H) )</th>
<th>( s_i(B) )</th>
<th>( s_i(H) )</th>
<th>( \lambda_i(H') )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0000</td>
<td>1.2</td>
<td>7.1</td>
<td>2.2725</td>
</tr>
<tr>
<td>-1.1869</td>
<td>1.4</td>
<td>3.8</td>
<td>-1.8652</td>
</tr>
<tr>
<td>0.47473 ± 1.4373i</td>
<td>1.9</td>
<td>2.6</td>
<td>0.31126 ± 1.4433i</td>
</tr>
<tr>
<td>-0.38127 ± 1.2286i</td>
<td>1.7</td>
<td>4.0</td>
<td>-0.51492 ± 0.77502i</td>
</tr>
</tbody>
</table>

Suppose now that the reduction to Hessenberg form was carried out in truncated 4-bit arithmetic. (This example can be generalized to \( n \) dimensions and \( (n - 2) \)-bit arithmetic.) The reduced matrix becomes

\[
H' = \begin{bmatrix}
0 & -2 & -1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & -1 \\
0 & 1 & 0 & 2 & -2 \\
0 & 0 & 1 & 4 & -4 \\
0 & 0 & 0 & 8 & -8 \\
0 & 0 & 0 & 0 & -8
\end{bmatrix}
\]

whose \((6, 5)\)-element differs slightly from the corresponding element of \( H \). Due to sensitivity to this difference, the eigenvalues \( \lambda_i(H') \) show little resemblance to those of \( B \).

We have shown that there exist matrices which cannot be stably reduced to Hessenberg form by means of Gaussian elimination in finite precision arithmetic. Householder transformations, however, provide unconditional stability.

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