Nonnegative Matrix Equations
Having Positive Solutions

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Abstract. Suppose \( \bar{A} \) is a nonnegative invertible matrix with a positive diagonal \( D = \text{Diag} (\bar{A}) > 0 \) and \( \bar{y} > 0 \) is a positive vector. Let \( A = D^{-1}\bar{A} \) and \( y = D^{-1}\bar{y} \). If \( 0 < 2y - Ay \), then \( 2y - Ay \preceq x \preceq y \), where \( x = A^{-1}y \).

Introduction. The inverse \( A^{-1} \) of a given nonnegative invertible matrix, \( A \), will usually contain negative elements; and hence for some \( y > 0 \) the solution vector \( x = A^{-1}y \) will have negative components. As suggested in the abstract there is no loss in generality in assuming \( \text{Diag} (A) = I \). The condition

\[
0 < 2y - Ay
\]

will be shown to imply \( 0 < x = A^{-1}y \) and to imply that \( A \) is diagonally similar to the diagonally dominant matrix \( Y^{-1}AY \).

Theorem. Suppose \( \bar{A} \) is a nonnegative invertible matrix with a positive diagonal \( D = \text{Diag} (\bar{A}) > 0 \) and \( \bar{y} > 0 \) is a positive vector. Let \( A = D^{-1}\bar{A} \) and \( y = D^{-1}\bar{y} \). If \( 0 < 2y - Ay \), then \( 2y - Ay \preceq x \preceq y \), where \( x = A^{-1}y \).

Proof. Let \( B = A - I \) then (1) implies \( 0 < (I - B)y \). We wish to show \( 2y - Ay \preceq x \preceq y \), i.e. \( (I - B)y \preceq (I + B)^{-1}y \preceq y \), i.e. \( (I - B)y \preceq (I - B^2)^{-1} (I - B)y \preceq y \). Let \( u \) be the positive vector \( u = (I - B)y \). We wish to show \( u \preceq (I - B^2)^{-1} u \preceq (I - B)^{-1} u \) which will hold provided \( (I - B^2)^{-1} \) and \( (I - B)^{-1} \) are nonnegative matrices.

These matrices will be nonnegative provided the corresponding matrix series converge, since

\[
I \preceq I + B^2 + B^4 + \cdots \preceq I + B + B^2 + \cdots
\]

implies \( I \preceq (I - B^2)^{-1} \preceq (I - B)^{-1} \).

And the series will converge provided the spectral radius of \( B \) satisfies \( \rho(B) < 1 \). To see that \( \rho(B) < 1 \), we let \( y = Ye \) where \( e \) is the vector having all its components equal to 1 and \( Y \) is the diagonal matrix corresponding to \( y \). Then, \( 0 < (I - B)y \) implies \( Y^{-1}BYe < e \) which implies \( \rho(B) = \rho(Y^{-1}BY) < 1 \).

Corollary. The inequality \( Y^{-1}BYe < e \) also implies that the matrix \( (I + Y^{-1}BY) = Y^{-1}(I + B)Y \) is diagonally dominant.

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