Finite-Difference Methods and the Eigenvalue Problem for Nonselfadjoint Sturm-Liouville Operators*

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Abstract. In this paper we analyze the convergence of a centered finite-difference approximation to the nonselfadjoint Sturm-Liouville eigenvalue problem

\[ L[u] = -a(x)u'' - b(x)u' + c(x)u = \lambda u, \quad 0 < x < 1, \]
\[ u(0) = u(1) = 0 \]

where \( L \) has smooth coefficients and \( a(x) \geq a_0 > 0 \) on \([0, 1]\). We show that the rate of convergence is \( O(\Delta x^2) \) as in the selfadjoint case for a scheme of the same accuracy. We also establish discrete analogs of the Sturm oscillation and comparison theorems. As a corollary we obtain the result

\[ \lim_{M \to \infty; \Delta x \to 0; (M+1)\Delta x = 1} \left( \sum_{p=1}^{M} \frac{\|V_p\|_2}{\Lambda_p} \right) < \infty \]

where \( \Delta x = 1/(M + 1) \) is the mesh size and \( \Lambda_p, V_p \) are the characteristic pairs of \( L \), the \( M \times M \) matrix which approximates \( L \), and \( V_p \) is normalized so that \( \|V_p\|_2 = 1 \).

1. Introduction. Many authors (e.g. [1], [6], [8], [9]) have studied the convergence of finite-difference methods for selfadjoint Sturm-Liouville eigenvalue problems. In this report we are concerned with the nonselfadjoint problem

\[ L[u] = -[a(x)u']' - b(x)u' + c(x)u = \lambda u, \quad 0 < x < 1, \]
\[ u(0) = u(1) = 0 \]

where \( a(x) \geq a_0 > 0 \), \( c(x) \geq 0 \), and \( b(x) \) are all smooth functions. This problem has an infinite sequence of positive [12, p. 37] and distinct [13, p. 212] eigenvalues

\[ 0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots \]

and a corresponding sequence of smooth eigenfunctions \( u^1(x), u^2(x), u^3(x), \cdots \) which we assume normalized so that

\[ \int_0^1 |u^p|^2 \, dx = 1, \quad p = 1, 2, \cdots. \]

Of course, as is well known, the transformation

\[ u(x) = \left[ \exp \left( -\frac{1}{2} \int_0^x \frac{b(t)}{a(t)} \, dt \right) \right] v(x) \]

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puts (1.1) into the selfadjoint form

\[ L[v] = -(av')' + (c + \frac{1}{2} b' + \frac{1}{4} (b^2/a)v = \lambda v , \]
\[ v(0) = v(1) = 0 . \]

However, we consider the direct approximation of (1.1) by means of the finite-difference equations

\[ -\frac{a_{k+1/2}(w_{k+1} - w_k) - a_{k-1/2}(w_k - w_{k-1})}{\Delta x^2} - \frac{b_k(w_{k+1} - w_{k-1})}{2\Delta x} + c_k w_k = \Lambda w_k , \quad k = 1, 2, \ldots, M , \]
\[ w_0 = w_{M+1} = 0 \]

where \( M \) is a large positive integer, \( \Delta x = 1/(M + 1) \) is the mesh spacing and the notation \( g_k \gamma \) is used for \( g(k \Delta x) \). Equivalently, we may write (1.5) as the finite-dimensional eigenvalue problem:

\[ LW = \Lambda W \]

where \( W \) is the \( M \) component vector

\[ W = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_M \end{bmatrix} \]

and \( L \) the \( M \times M \) tridiagonal matrix

\[ L = \frac{1}{\Delta x^2} \begin{bmatrix} \alpha_1 & \beta_1 & 0 & & & \\ \gamma_2 & \alpha_2 & \beta_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \beta_{M-1} \\ 0 & & & & \gamma_M & \alpha_M \end{bmatrix} \]

with

\[ \alpha_k = [a_{k+1/2} + a_{k-1/2}] + c_k \Delta x^2 \beta_k = -[a_{k+1/2} + b_k \Delta x/2] \quad \text{and} \]
\[ \gamma_k = [b_k \Delta x/2 - a_{k-1/2}] \quad k = 1, 2, \ldots, M . \]

We will show that the latter procedure preserves the rate of convergence, namely \( O(\Delta x^2) \), which obtains in the selfadjoint case for a scheme of the same accuracy, (see [6]). This is Theorem 1.

The matrix \( L \) defined above will be shown to be similar to an oscillation matrix, by means of a diagonal transformation \( \bar{D} \). Using the basic theorem on oscillation matrices, (see [4], [5]) and the fact that the entries of \( \bar{D} \) alternate in sign, one immediately has a discrete analog of the Sturm Oscillation Theorem [13, p. 212, Theorem
2.1] namely $L$ has positive distinct eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_M$ and if $W^j$ is an eigenvector belonging to $\lambda_j$ then $W^j$ has exactly $j - 1$ nodes** in $0 < x < 1$. Moreover the nodes of successive eigenvectors alternate.***

We will also show the following

**Theorem 2.** Let $V^j$ be an eigenvector of $L$ corresponding to $\lambda_j$ and let $\delta_{\text{max}}(V^j)$ be the maximum distance between successive nodes of $V^j$. Then there exists an integer $j_0$, independent of $M$, and a positive constant $K_1$, such that for $j_0 \leq j \leq M$, we have

\begin{equation}
\delta_{\text{max}}(V^j) \leq K_1(\lambda_j)^{-1/2}.
\end{equation}

In the continuous case, the estimate (1.9) is usually obtained as a corollary to the Sturm Comparison Theorem [14, p. 224]. We will base the proof of Theorem 2 on a discrete maximum principle. We remark that in the continuous case, proofs of the oscillation and comparison theorems, based on a maximum principle, have been given by Protter and Weinberger in their recent book. See [12].

### 2. Symmetrization of the Discrete Problem.

**Definitions.** For any two $M$ vectors $X$, $Y$ define their scalar products by

$$\langle X, Y \rangle = \Delta x \sum_{k=1}^{M} x_k y_k$$

and let

$$\|X\|_2 = \left( \Delta x \sum_{k=1}^{M} |x_k|^2 \right)^{1/2}$$

be the corresponding norm.

If $A$ is an $M \times M$ matrix then we define

$$\|A\|_2 = \sup_{X \neq 0} \frac{\|AX\|_2}{\|X\|_2}.$$ 

**Lemma 1.** Fix $\Delta x > 0$ sufficiently small. Then there exists a nonsingular, positive, diagonal matrix $D$ such that $D^{-1}LD = L$ is a real symmetric matrix. Moreover, $\|D\|_2$, $\|D^{-1}\|_2$ remain bounded as $M \to \infty$, $\Delta x \to 0$, $(M + 1) \Delta x = 1$.

**Proof.** We construct such a matrix. Let

$$D = \begin{bmatrix}
d_1 & 0 \\
d_2 & 0 \\
& \ddots & \ddots \\
& & \ddots & 0 \\
& & & d_M
\end{bmatrix}$$

where $d_j \neq 0, j = 1, \cdots, M$.

**As in [5], a “node” of $V^j$ is a point where the graph of $V^j$, (i.e. the graph of the piecewise-linear function obtained from $V^j$ by linear interpolation) intersects the $x$-axis.**

*** These observations about $L$ are not new. (See Sinden [17] and Varga [18, p. 206].)
and \( d_1 = 1 \) and let \( \hat{L} = D^{-1}LD = (l_{ij}) \).

Since we require \( \hat{L} = \hat{L}^T \) we must have

\[
d_i^{-1}l_{i,j}d_j = d_j^{-1}l_{i,j}d_i \quad \text{where} \quad L = (l_{ij}) .
\]

Further, since \( l_{ij} = 0 \) for \( j > i + 1, j < i - 1 \), the \( d_j \)'s must be determined so that

\[
d_i^2 = \frac{l_{i,i-1}}{l_{i-1,i}} d_{i-1}^2, \quad i = 2, \ldots, M .
\]

Starting from \( d_1 = 1 \), we may solve recursively to obtain

\[
d_i^2 = \prod_{k=1}^{i-1} \left( \frac{\gamma_{k+1}}{\beta_k} \right), \quad i = 2, \ldots, M
\]

and, since \( \gamma_k, \beta_k < 0 \) for sufficiently small \( \Delta x \), \( d_i^2 > 0 \) if \( \Delta x \) is small enough.

With \( D \) constructed as above, we have

\[
\hat{L} = \frac{1}{\Delta x^2} \begin{bmatrix}
\alpha_1 & -\left(\gamma_2\beta_1\right)^{1/2} & 0 \\
& \ddots & \ddots \\
& & \ddots & -\left(\gamma_M\beta_{M-1}\right)^{1/2} \\
& & & \alpha_M
\end{bmatrix}
\]

and we must show that \( \|D\|_2, \|D^{-1}\|_2 \) remain bounded as \( M \to \infty \). Let

\[
Q_i = \prod_{k=1}^{i-1} \left( 1 - \frac{b_{k+1}\Delta x}{2a_{k+1/2}} \right)
\]

and

\[
P_i = \prod_{k=1}^{i-1} \left( 1 + \frac{b_k\Delta x}{2a_{k+1/2}} \right)
\]

then \( d_i^2 = Q_i/P_i \). Now for sufficiently small \( \Delta x \),

\[
\log \left( 1 - \frac{b_{k+1}\Delta x}{2a_{k+1/2}} \right) = -\frac{b_{k+1}\Delta x}{2a_{k+1/2}} + O(\Delta x^2)
\]

so that

\[
\log Q_i = -\Delta x \sum_{k=1}^{i-1} b_k + \Delta x \sum_{k=1}^{i-1} O(\Delta x).
\]
Hence,
\[ \lim_{\Delta x \to 0, i \to \infty; i \Delta x \to \xi} [\log Q_i] = -\frac{1}{2} \int_0^\xi \frac{b(t)}{a(t)} \, dt. \]

Similarly,
\[ \lim_{\Delta x \to 0, i \to \infty; i \Delta x \to \xi} [\log P_i] = \frac{1}{2} \int_0^\xi \frac{b(t)}{a(t)} \, dt. \]

Consequently,
\[ \lim d_i = \left[ \exp \left( -\frac{1}{2} \int_0^\xi \frac{b(t)}{a(t)} \, dt \right) \right] \leq K_0 < \infty \]

which shows both \( ||D||_2, ||D^{-1}||_2 \) remain bounded as \( \Delta x \to 0, M \to \infty, (M + 1) \Delta x = 1 \).

**Lemma 2.** For \( \Delta x \) sufficiently small, the eigenvalues of \( L \) are strictly positive and they remain bounded away from zero as \( M \to \infty, \Delta x \to 0, (M + 1) \Delta x = 1 \).

**Proof.** For \( \Delta x \) sufficiently small, \( \gamma_k, \beta_k < 0 \). Hence if \( L = (l_{ij}) \) and \( \Omega_i = \sum_{j \neq i} |l_{ij}| \), then
\[ \Omega_i = \frac{(a_{i+1/2} + a_{i-1/2})}{\Delta x^2} \]
and \( l_{ii} = \frac{(a_{i+1/2} + a_{i-1/2})}{\Delta x^2} + c_i \geq \Omega_i \) since \( c_i \geq 0 \).

By Gershgorin’s theorem, [7], the eigenvalues of \( L \) lie in the union of the discs \( |z - l_{ii}| \leq \Omega_i \) in the complex plane. Hence if \( \Lambda \) is an eigenvalue of \( L \), then \( \Lambda \geq 0 \) since \( \Lambda \) is real.

Now let \( l_h \) be the finite-difference operator corresponding to \( -L \), i.e.
\[ [l_h v]_k = -\left[ \frac{(a_{k+1/2} + a_{k-1/2}) + c_k \Delta x^2}{\Delta x^2} v_k + \frac{a_{k+1/2} + b_k \Delta x/2}{\Delta x^2} v_{k+1} \right] \]
\[ + \left[ \frac{a_{k+1/2} - b_k \Delta x/2}{\Delta x^2} \right] v_{k-1}. \]

Then, for sufficiently small \( \Delta x \), \( l_h \) is of positive type [3, p. 181] and so satisfies the discrete maximum principle [16, p. 23, Lemma 2.3]. Consequently [16, p. 108, Theorem 7.1] if \( w(k \Delta x), k = 0, 1, \ldots, M + 1 \) is an arbitrary real-valued mesh function, there exists positive constants \( K \) and \( \delta \) such that if \( 0 < \Delta x < \delta \),
\[ (2.2) \quad \|w\|_\infty \leq \max_k |w_k| \leq \max \{ |w_0|, |w_{M+1}| \} + K \|l_h w\|_\infty. \]

Now let \( V = \{v_k\}_{k=1}^M \) be an eigenvector of \( L \) corresponding to \( \Lambda \). We may assume \( V \) to be real. Defining \( v_0 = v_{M+1} = 0, L V = \Lambda V \) is equivalent to
\[ (2.3) \quad [l_h v]_k = -\Lambda v_k, \quad k = 1, \ldots, M. \]

Hence, using (2.2) and the fact that \( \Lambda \geq 0 \),
\[ \|v\|_\infty \leq K \|l_h v\|_\infty = \Lambda K \|v\|_\infty \]
i.e. \( \Lambda \geq 1/K > 0 \). Q.E.D.

**Corollary.** Let \( \Gamma \) be the \( M \times M \) matrix given by
and let $\hat{L}$ be defined by (2.1), then $\Gamma^{-1} \hat{L} \Gamma$ is an oscillation matrix.

Proof. $\Gamma^{-1} \hat{L} \Gamma$ is a positive-definite real symmetric matrix with positive elements along the first super and sub diagonals. The proof now follows from a theorem of Gantmacher and Krein [4, p. 103].

3. Convergence of the Characteristic Pairs of $L$. Let $0 < \Lambda_1 < \Lambda_2 < \cdots < \Lambda_M$ be the eigenvalues of $L$. Fix a positive integer $p$ and let $V^p(\Delta x)$ be the eigenvector corresponding to $\Lambda_p(\Delta x)$, normalized so that $\|V^p\|_2 = 1$. Let $\tilde{V}^p$ be the continuous piecewise-linear function, vanishing at $x = 0, 1$, and which, in the interior of $[0, 1]$, is obtained from $V^p$ by linear interpolation. Consider the families $\{\Lambda_p(\Delta x)\}$, $\{\tilde{V}^p(\Delta x)\}$ as the mesh size $\Delta x \to 0$.

Using Lemma 1 and considering the symmetrized problem, one can give a direct proof of uniform convergence of $\tilde{V}^p$ to $u^p$ and $\Lambda_p$ to $\lambda_p$ as $\Delta x \to 0$. (See [2].) This method of proof is based on the compactness of the family $\{\tilde{V}^p(\Delta x)\}$ in $C[0, 1]$ and has been used several times by Parter (see [9], [10], [11]) but it does not immediately yield estimates on the rates of convergence. Nevertheless we will make use below (see Eq. 3.8 (1)) of the fact that $\Lambda_p \to \lambda_p$ together with Lemma 1 above to obtain these estimates. The proof given below is a modification of that given by Gary in [6] for the selfadjoint case.

**Theorem 1.** Let $\Lambda_p, V^p$ be characteristic pairs of $L$ with $\|V^p\|_2 = 1$. Let $D$ be the diagonal matrix of Lemma 1. Let $u^p$ be an eigenfunction of $\mathcal{L}$ corresponding to $\lambda_p$ and let $U^p$ be the $M$ vector obtained from $u^p$ by mesh-point evaluation. Assume $u^p(x)$ normalized so that

$$(3.1) \quad \|D^{-1}U^p\|_2 = \|D^{-1}V^p\|_2$$

then as $\Delta x \to 0$, we have

$$(3.2) \quad |\lambda_p - \Lambda_p| \leq K \Delta x^2,$$

$$(3.3) \quad \|U^p - V^p\|_2 \leq K_1 \Delta x^2$$

where $K, K_1$ are positive constants depending only on $p$.

Proof. Because the difference scheme in (1.5) is properly centered and we assume sufficient smoothness of $u^p$ and the coefficients of $\mathcal{L}$, we have at the mesh points,

$$(3.4) \quad \mathcal{L}[u^p] = LU^p + \tau = \lambda_p U^p$$

where $\tau$ is the “truncation” error and

$$(3.5) \quad \|\tau\|_2 \leq K(p) \Delta x^2$$

where $K$ is a constant.

Let $\hat{L} = D^{-1}LD$ have orthonormal eigenvectors $X^1, X^2, \cdots, X^M$ and write $U^p$ as a linear combination of the $DX^p$s:
(3.6) \[ U^p = \sum_{j=1}^{M} \sigma_j DX^j \]

so that

\[ LU^p = \sum_{j=1}^{M} \sigma_j LDX^j = \sum_{j=1}^{M} \sigma_j \Lambda_j DX^j \]

then

\[ \tau = (\lambda_p - L)U^p = \sum_{j=1}^{M} \sigma_j (\lambda_p - \Lambda_j) DX^j \]

and

(3.7) \[ \sum_{j \neq p}^{M} \sigma_j^2 |\lambda_p - \Lambda_j|^2 = \|D^{-1}r\|_2^2 \leq \|D^{-1}\|_2^2 \|r\|_2^2 \leq K_1(p) \Delta x^4 \]

where \( K_1 \) is a constant.

Now, the eigenvalues of \( L \) are distinct and converge to the corresponding distinct eigenvalues of \( \Omega \). It follows that

(3.8) \[ \inf_{j \neq p} \{|\lambda_p - \Lambda_j|\} \geq \omega_0 > 0 \]

for all sufficiently small \( \Delta x \). Hence, on using (3.7),

(3.9) \[ \sum_{j \neq p}^{M} \sigma_j^2 \leq K_1 \Delta x^4. \]

From (3.9), (3.6) we obtain

(3.10) \[ \sigma_p^2 = \|D^{-1}U^p\|_2^2 + O(\Delta x^4) \geq \omega_1 > 0 \]

for all sufficiently small \( \Delta x \).

Thus

(3.11) \[ |\lambda_p - \Lambda_p| \leq K_2(p) \Delta x^2. \]

Since \( V^p = \beta DX^p \) for some \( \beta \) and \( \|X^p\|_2 = 1 \) we have

\[ |\beta| = \|D^{-1}V^p\|_2. \]

On taking square roots in (3.10), we have

\[ \sigma_p = \|D^{-1}U^p\|_2 + O(\Delta x^4) \]

and we may assume that \( \sigma_p \) and \( \beta \) have the same sign; hence using (3.1),

(3.12) \( (\sigma_p - \beta) = O(\Delta x^4) \).

Writing \( U^p - V^p = \sum_{j \neq p} \sigma_j DX^j + (\sigma_p - \beta) DX^p \) we have

(3.13) \[ \|D^{-1}(U^p - V^p)\|_2^2 = \sum_{j \neq p} \sigma_j^2 + (\sigma_p - \beta)^2 = O(\Delta x^4) \]

i.e.

(3.14) \[ \|U^p - V^p\|_2^2 \leq \|D\|_2^2 \|D^{-1}(U^p - V^p)\|_2^2 \leq K_3(p) \Delta x^4. \] Q.E.D.

Notice that the above inequality also implies uniform convergence at the rate of \( O(\Delta x^{3/2}) \).
4. Proof of Theorem 2.

**Lemma 3.** Let $0 < A_1 < \cdots < A_M$ be the eigenvalues of $L$. Then there exists a positive integer $j_0$, independent of $M$, such that for $j_0 \leq j \leq M$ we have

$$K_1 j^2 \pi^2 \leq A_j \leq K_2 j^2 \pi^2, \quad K_1, K_2 \text{ positive constants}.$$  

**Proof.** In the selfadjoint case this result may be found in Bückner [1]. In the present more general case we will need to estimate the off-diagonal elements of the matrix $\hat{L}$ in Lemma 1.

With the notation of (1.8) let

$$q_k^2 = \gamma_k + b_k \Delta x = (a_{k+1/2} + b_k \Delta x/2), \quad k = 1, \ldots, M - 1.$$  

Since $b(x) \in C^1[0, 1]$, we have by the mean-value theorem,

$$q_k^2 = (a_{k+1/2})^2 [1 - 2 \mu_k \Delta x^2 + O(\Delta x^3)]$$

where $2 \mu_k = [b_k^2 + 2a_{k+1/2} b'(\xi_k)] / 4a_{k+1/2}$ for some $\xi_k$ such that $k \Delta x < \xi_k < (k + 1) \Delta x$. Hence on taking square roots

$$q_k = a_{k+1/2}[1 - \mu_k \Delta x^2 + O(\Delta x^3)], \quad k = 1, \ldots, M - 1.$$  

We now proceed to estimate the quadratic form $\langle X, \hat{L}X \rangle$ where $X$ is any complex $M$ vector of norm 1. Defining $x_0 = x_{M+1} = 0$, and using (4.3), we may write

$$\langle X, \hat{L}X \rangle = \Delta x \sum_{k=0}^M \frac{|x_k - x_{k+1}|^2}{\Delta x^2} + \Delta x \sum_{k=1}^M c_k |x_k|^2$$

$$+ 2 \Delta x \sum_{k=0}^M \mu_k a_{k+1/2} x_k x_{k+1} + O(\Delta x) \Delta x \sum_{k=0}^M x_k x_{k+1}.$$  

Now let $0 < a_0 \leq a(x) \leq a_1$ on $[0, 1]$ and let

$$\|c\|_\infty = \max_k |c_k|, \quad \|\mu\|_\infty = \max_k |\mu_k|.$$  

We have

$$\langle X, \hat{L}X \rangle \leq a_1 \Delta x \sum_{k=0}^M \frac{|x_{k+1} - x_k|^2}{\Delta x^2} + \|c\|_\infty + 2a_1 \|\mu\|_\infty + O(\Delta x)$$

and

$$\langle X, \hat{L}X \rangle \geq a_0 \Delta x \sum_{k=0}^M \frac{|x_{k+1} - x_k|^2}{\Delta x^2} - 2a_1 \|\mu\|_\infty - O(\Delta x).$$

Let $H$ be the tridiagonal $M \times M$ matrix defined by

$$H = \frac{1}{\Delta x^2} \begin{bmatrix} 2 & -1 & 0 \\
\vdots & \ddots & \ddots \\
-1 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & -1 & 2 \end{bmatrix}.$$
It is easily verified that

\begin{equation}
\langle X, HX \rangle = \Delta x \sum_{k=0}^{M} \frac{|x_{k+1} - x_k|^2}{\Delta x^2}
\end{equation}

and that the eigenvalues $\theta_j$, $j = 1, \cdots, M$, of $H$, arranged in increasing order, are given by

\begin{equation}
\theta_j = \frac{4}{\Delta x^2} \sin^2 \frac{j\pi \Delta x}{2}, \quad j = 1, \cdots, M.
\end{equation}

Inserting (4.9) into (4.6), (4.7) and using the maximum principle for the eigenvalues of real symmetric matrices shows that

\begin{equation}
a_k \theta_j - 2\|\mu\|_\infty - |O(\Delta x)| \leq \Lambda_j \leq a_k \theta_j + \|c\|_\infty + 2a_1\|\mu\|_\infty + |O(\Delta x)|.
\end{equation}

Using (4.10) and an elementary calculation, the proof follows from (4.11).

\textit{Proof of Theorem 2.} Let

\begin{equation}
W^j = \begin{bmatrix} w_1^j \\ \vdots \\ w_M^j \end{bmatrix}
\end{equation}

be an eigenvector of $L$ corresponding to $\Lambda_j$. Then $W^j$ satisfies the difference equations:

\begin{equation}
\begin{aligned}
&\left[-2 + \frac{(c_k - \Lambda_j)\Delta x^2}{\omega_k}\right]w_k^j + \left[\frac{a_{k+1/2} + b_k\Delta x/2}{\omega_k}\right]w_{k+1}^j \\
&+ \left[\frac{a_{k-1/2} - b_k\Delta x/2}{\omega_k}\right]w_{k-1}^j = 0, \quad k = 1, \cdots, M
\end{aligned}
\end{equation}

where $w_0^j = w_{M+1}^j = 0$ and $\omega_k = \frac{1}{2} (a_{k+1/2} + a_{k-1/2})$.

Let

\begin{equation}
\bar{\alpha}_k = -\left[2 + \frac{(c_k - \Lambda_j)\Delta x^2}{\omega_k}\right], \quad \bar{\beta}_k = \left[\frac{a_{k+1/2} + \frac{1}{2} b_k\Delta x}{\omega_k}\right],
\end{equation}

and let $A$ be the tridiagonal $M \times M$ matrix

\begin{equation}
A = \begin{bmatrix}
\bar{\alpha}_1 & \bar{\beta}_1 \\
& \ddots & \ddots \\
\bar{\gamma}_2 & \ddots & \ddots \\
& \ddots & \ddots & \ddots \\
& \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & \ldots & \ldots & \bar{\alpha}_M \\
\end{bmatrix}.
\end{equation}
Then we may write (4.12) as

(4.14) \[ AW^j = 0 \]

or equivalently

(4.15) \[ (P^{-1}AP)P^{-1}W^j = 0 \]

if \( P \) is any nonsingular matrix.

Choose \( P \) to be the diagonal matrix

(4.16) \[ P = \begin{bmatrix} p_1 & & & \\ & \ddots & & \\ & & \ddots & \\ 0 & \cdots & \cdots & p_M \end{bmatrix} \]

where \( p_1 = 1 \) and \( p_i^2 = \prod_{k=1}^{i-1} (\gamma_{k+1}/\beta_k) \), \( i = 2, \cdots, M \).

For all sufficiently small \( \Delta x, p_i > 0 \) and as in Lemma 1, \( P \) symmetrizes \( A \). Let \( \sigma_k = (\gamma_{k+1}/\beta_k)^{1/2} \), then

(4.17) \[ P^{-1}AP = \begin{bmatrix} \tilde{\alpha}_1 & \sigma_1 & & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & \sigma_{M-1} \\ 0 & \cdots & \cdots & \tilde{\alpha}_M \end{bmatrix} \]

Observe that by the mean-value theorem

(4.18) \[ \omega_k - \omega_{k+1} = (a_{k+1/2})^2 [1 + O(\Delta x^2)] \quad \text{as} \quad \Delta x \to 0. \]

Also if \( b(x) \in C^1[0, 1] \),

(4.19) \[ (\gamma_{k+1}/\beta_k) = \left( \frac{(a_{k+1/2})^2 + a_{k+1/2}(b_k - b_{k+1}) \Delta x}{\omega_k \omega_{k+1}} \right) \]

\[ = \left( \frac{(a_{k+1/2})^2 [1 + O(\Delta x^2)]}{(a_{k+1/2})^2 [1 + O(\Delta x^2)]} \right) \quad \text{as} \quad \Delta x \to 0. \]

Hence,

(4.20) \[ \sigma_k = (\gamma_{k+1}/\beta_k)^{1/2} = 1 + O(\Delta x^2) \quad \text{as} \quad \Delta x \to 0. \]

Let \( V = P^{-1}W^j \) and write the system (4.15) as

(4.21) \[ \begin{bmatrix} 2 + \left( \frac{c_k - \lambda_j}{\omega_k} \right) \Delta x^2 \end{bmatrix} v_k + \sigma_k v_{k+1} + \sigma_k v_{k-1} = 0, \]

\[ v_0 = v_{M+1} = 0, \quad k = 1, \cdots, M. \]

Let \( K_1 \) and \( K_2 \) be the constants in Lemma 3 and define
\( \beta_j^2 = \Lambda_j/K_2 \).

Let \( y(x) = \sin \beta_j x \). Then \( y_k = y(k \Delta x) \) satisfies the difference equations:

\[
-2 - \mu_j \Delta x^2 \right] y_k + y_{k+1} + y_{k-1} = 0, \quad k = 1, 2, \ldots
\]

where

\[
\mu_j = \frac{4}{\Delta x^2} \sin \frac{\beta_j \Delta x}{2}.
\]

The distance between successive zeros of \( y(x) \) is \( \pi/\beta_j = (K_2 \pi^2/\Lambda_j)^{1/2} \geq 1/j \) for \( j \) large enough by Lemma 3.

Let \( v(x) \) be the piecewise-linear function corresponding to "graph" of vector \( V = P^{-1}W' \). Define the auxiliary function \( z(x) \) by

\[
z(x) = \frac{y(x)}{v(x)} \quad \text{whenever } v(x) \neq 0.
\]

We proceed to estimate the distance between successive nodes of \( v(x) \) by investigating the difference equation satisfied by \( z(x) \).

We may assume that \( \delta_{\text{max}}(V) > 3 \Delta x \); if not, \( \delta_{\text{max}}(V) \leq 3 \Delta x \), then in particular, \( \delta_{\text{max}}(V) \leq 3/(M + 1) < 3/j \leq 3\pi(K_2/\Lambda_j)^{1/2} \) for all sufficiently large \( j \). If \( \delta_{\text{max}} > 3 \Delta x \), then there exists a set \( N \) of consecutive mesh points, containing at least three members on which \( v(x) \) is strictly positive (or strictly negative). Let \( N' \) be \( N \) minus the two end points of \( N \). Since \( z_k = y_k/v_k \) for \( k \in N' \),

\[
[lz\right]_k = \left[ \frac{(2 - \mu_j \Delta x^2)\sigma_k}{2 + (c_k - \Lambda_j) \Delta x^2/\omega_k} \right] z_k + v_{k+1}z_{k+1} + v_{k-1}z_{k-1} = 0, \quad k \in N'.
\]

We now show that for all sufficiently large \( j \), the difference operator \( l_z \) (or \(-l_z\) if \( v \) is strictly negative) occurring in (4.25) is of positive type, and hence satisfies the discrete maximum principle:

It is sufficient to show that if \( j \) is sufficiently large,

\[
\frac{[2 - \mu_j \Delta x^2] \sigma_k}{2 + (c_k - \Lambda_j) \Delta x^2/\omega_k} \geq 1, \quad \text{if } k \in N'.
\]

From (4.24) we have \( \mu_j \leq \Lambda_j/K_2 \leq \Lambda_j/2a_1 \) if \( K_2 \) is chosen so that \( K_2 \geq 2a_1 \), where \( a_1 \) is an upper bound for \( a(x) \) on \([0, 1]\). Hence,

\[
(2 - \mu_j \Delta x^2)\sigma_k = 2 - \mu_j \Delta x^2 + O(\Delta x^2)
\]

since \( \mu_j \Delta x^2 \leq 4 \) and \( \sigma_k = 1 + O(\Delta x^2) \). Now,

\[
2 - \mu_j \Delta x^2 + O(\Delta x^2) \geq 2 - \Lambda_j \Delta x^2/K_2 + O(\Delta x^2) \geq 2 - \Lambda_j \Delta x^2/2\omega_k + O(\Delta x^2) = 2 + \frac{(c_k - \Lambda_j) \Delta x^2}{\omega_k} + \frac{(\Lambda_j - 2c_k) \Delta x^2}{2\omega_k} + O(\Delta x^2)
\]

i.e.

\[
(2 - \mu_j \Delta x^2)\sigma_k \geq 2 + (c_k - \Lambda_j) \Delta x^2/\omega_k
\]
if \( j \) is sufficiently large, since we assume \( c(x) \) is bounded.

Furthermore, \( 2 + (c_k - \Lambda_j) \Delta x^2/\omega_k \) is positive for \( k \in N' \) since \( v_k, v_{k+1}, v_{k-1} \) have the same sign, on using (4.21). Thus (4.26) is satisfied.

Suppose now that \( z(x) \) has two zeros in the interval spanned by \( N \). At any mesh point lying between the two zeros we must have \( z(x) = 0 \) by the maximum principle. Since \( z(x) = 0 \) if and only if \( y(x) = 0 \), this means that the distance between successive zeros of \( y(x) \) is \( \leq \Delta x = 1/(M + 1) \). However, as already noted, this distance is \( \geq 1/j \) and \( j \leq M \).

Thus \( y(x) \) has at most one zero in the interval spanned by \( N \). Hence the maximum distance between successive nodes of \( v(x) \) must be less than or equal to \( \pi/\beta_j + 2\Delta x \). Since \( \Lambda_j = O(1/\Delta x^2) \), we have

\[
(4.29) \quad \delta_{\text{Max}}(V) \leq K(\Lambda_j)^{-1/2}.
\]

A similar estimate is valid for the eigenvector \( W^j \) of \( L \) since \( W^j = PV \) and \( P \) is a positive diagonal matrix. Q.E.D.

**Corollary 1.** Let the eigenvectors \( \{V^p\} \) of \( L \) be normalized so that \( \|V^p\|_2 = 1 \).

Then there exists a constant \( K \) and an integer \( p_0 \), both independent of \( M \) such that if \( p_0 \leq p \leq M \),

\[
\|V^p\|_\infty = \max_{k=1 \cdots M} |v_k^p| \leq K^{1/2}.
\]

**Proof.** Let \( W^p \) be the normalized eigenvector of \( \hat{L} = D^{-1}LD \) corresponding to \( \Lambda_p \). Since \( W^p = D^{-1}V^p/\|D^{-1}V^p\|_2 \) and \( D^{-1} \) is a positive diagonal matrix, the distance between successive nodes of \( W^p \) satisfies an estimate similar to (4.29). Since \( W^p \) is normalized we have

\[
(4.30) \quad \|V^p\|_\infty = \max_{k=1 \cdots M} |v_k^p| \leq K^{1/2}.
\]

Hence, using inequality (4.7) in the proof of Lemma 3, we get,

\[
(4.31) \quad \langle W^p, \hat{L}W^p \rangle = \Lambda_p.
\]

for all sufficiently large \( p \).

Let \( r, s \) be any two positive integers with \( 1 \leq s < r \leq M \). Then,

\[
|w_{r}^p - w_{s}^p| = \left| \Delta x \sum_{k=s}^{r-1} \frac{w_{k+1}^p - w_k^p}{\Delta x} \right|
\]

\[
\leq \left[ (r - s) \Delta x \right]^{1/2} \left( \Delta x \sum_{k=0}^{M} \frac{|w_{k+1}^p - w_k^p|^2}{\Delta x^2} \right)^{1/2}
\]

\[
\leq \left[ (r - s) \Delta x \right]^{1/2} (2\Lambda_p/a_0)^{1/2}
\]

on using Schwarz’s inequality and (4.32). Now choose \( r \) so that \( |w_r^p| = \|W^p\|_\infty > 0 \) and let \( s \) be the integer nearest \( r \) with the property that \( w_rw_s^p \leq 0 \). (\( s \) need not necessarily be less than \( r \).) We then have for sufficiently large \( p \), by Theorem 2,

\[
(4.33) \quad |(r - s) \Delta x| < 2\delta_{\text{Max}}(W^p) \leq K'(\Lambda_p)^{-1/2}.
\]

Hence using (4.33), (4.34)
\[ \| W^p \|_\infty \leq |w_{r}^p - w_{s}^p| \leq \left[ (r - s) \Delta x \right]^{1/2} (2 \Lambda_p / a_0)^{1/2} \]

\[ \leq K''(\Lambda_p)^{1/4} \]

for sufficiently large \( p \) and the proof follows from Lemma 3.

Remark. The estimate (4.30) was obtained by Bückner [1] in the selfadjoint case using an elementary device. It would be interesting to know whether or not the discrete eigenvectors display this growth as \( M \to \infty \). In the case of the analytic problem (1.1) it is known (see [15, p. 334]) that the normalized eigenfunctions are uniformly bounded in the supremum norm.

Corollary 2. Let \( \{V^p\}_{p=1}^M \) be the eigenvectors of \( L \) normalized so that \( \|V^p\|_2 = 1 \), \( p = 1, \ldots, M \). Then,

\[ \limsup_{M \to \infty; \Delta x \to 0; (M+1)\Delta x = 1} \left( \sum_{p=1}^M \frac{\|V^p\|_\infty}{\Lambda_p} \right) < \infty. \]

Proof. This follows immediately from Lemmas 2, 3, and Corollary 1.

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