Table for Third-Degree Spline Interpolation
With Equally Spaced Arguments*

By T. N. E. Greville

Abstract. A table is given to facilitate the calculation of the parameters of the interpolating third-degree natural spline function for \( n \) given data points \( (n > 2) \) with equally spaced abscissas. The use of the table is described and the correctness of the algorithm is demonstrated.

1. Introduction. Given a set of \( n \) real numbers \( x_1 < x_2 < \cdots < x_n \) called "knots," a spline function of degree \( m \) having the knots \( x_j \) is defined to be a function \( S(x) \) satisfying the following two conditions:

(1) In each interval \( (x_j, x_{j+1}) \) \( (j = 0, 1, \cdots, n; x_0 = -\infty, x_{n+1} = \infty) \), \( S(x) \) is given by some polynomial of degree \( m \) (or less).

(2) The polynomial arcs which represent the function in successive intervals join smoothly in the sense that \( S(x) \) and its derivatives of order 1, 2, \( \cdots, m - 1 \) are continuous over \( (-\infty, \infty) \).

A spline function of odd degree \( 2k - 1 \) is called a "natural" spline function if it satisfies the further condition:

(3) In each of the two intervals \( (-\infty, x_1) \) and \( (x_n, \infty) \) \( S(x) \) is represented by a polynomial of degree \( k - 1 \) or less (in general, not the same polynomial in the two intervals).

It is well known [1] that given any set of \( n \) data points \( (x_j, y_j) \) with distinct abscissas, and an integer \( k \leq n \), there is a unique natural spline function \( s(x) \) of degree \( 2k - 1 \), having its knots limited to the abscissas \( x_j \), that also interpolates the given data points, in the sense that \( s(x_j) = y_j \) \( (j = 1, 2, \cdots, n) \). Moreover, in the class of continuous functions \( f(x) \) with continuous derivatives of order 1, 2, \( \cdots, k \) on \( (-\infty, \infty) \), this natural spline interpolating function is the "smoothest" interpolating function for the given data points, in the sense that the integral

\[
\int_a^b [f(x)]^2 \, dx
\]

(for any \( a, b \) such that \( a \leq x_1 \) and \( b \geq x_n \)) is smallest.

Third-degree spline functions (i.e., \( k = 2 \)) have been much more widely used than those of any other degree, and an algorithm is given in [1] for obtaining the third-degree interpolating natural spline function for any set of (2 or more) given data points with distinct abscissas. This algorithm involves the solution of an \( (n - 2) \times (n - 2) \) tridiagonal system of linear equations.

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If the abscissas of the data points are equally spaced, substantial simplification is possible, and the parameters of the third-degree interpolating natural spline function can be obtained explicitly, by the use of the table contained in this report, without the necessity of solving a system of equations.

2. Use of the Table. It is assumed that suitable changes of origin and scale have been made, if necessary, so that \( x_j = j \) \( (j = 1, 2, \cdots, n) \). On this assumption \( s(x) \) can be expressed [1] in the form

\[
s(x) = s(1) + (x - 1)d + \sum_{j=1}^{n} c_j(x - j)^3,
\]

where the truncated power function \( z^+ \) is given by

\[
z^+ = \begin{cases} 
  z^3 & (z \geq 0) \\
  0 & (z < 0) 
\end{cases}
\]

The coefficients \( d \) and \( c_j \) are to be determined.

**Table 1**

<table>
<thead>
<tr>
<th>( j )</th>
<th>( \alpha_j )</th>
<th>( \beta_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>-6</td>
<td>-4</td>
</tr>
<tr>
<td>4</td>
<td>24</td>
<td>15</td>
</tr>
<tr>
<td>5</td>
<td>-90</td>
<td>-56</td>
</tr>
<tr>
<td>6</td>
<td>-336</td>
<td>209</td>
</tr>
<tr>
<td>7</td>
<td>-1254</td>
<td>-780</td>
</tr>
<tr>
<td>8</td>
<td>4680</td>
<td>2911</td>
</tr>
<tr>
<td>9</td>
<td>-17466</td>
<td>-10864</td>
</tr>
<tr>
<td>10</td>
<td>65184</td>
<td>40545</td>
</tr>
<tr>
<td>11</td>
<td>-2 43270</td>
<td>-1 51316</td>
</tr>
<tr>
<td>12</td>
<td>9 07896</td>
<td>5 64719</td>
</tr>
<tr>
<td>13</td>
<td>-33 88314</td>
<td>-21 0750</td>
</tr>
<tr>
<td>14</td>
<td>126 45360</td>
<td>78 65521</td>
</tr>
<tr>
<td>15</td>
<td>-471 93126</td>
<td>-293 54524</td>
</tr>
<tr>
<td>16</td>
<td>1761 27144</td>
<td>1095 32575</td>
</tr>
<tr>
<td>17</td>
<td>-6573 15450</td>
<td>-4088 55776</td>
</tr>
<tr>
<td>18</td>
<td>24531 34656</td>
<td>15258 70529</td>
</tr>
<tr>
<td>19</td>
<td>-91552 23174</td>
<td>-56946 26340</td>
</tr>
<tr>
<td>20</td>
<td>3 41677 58040</td>
<td>2 12526 34831</td>
</tr>
</tbody>
</table>

The table can be continued by means of the following relations (the first of which does not hold for \( j = 3 \)):

\[
\begin{align*}
\alpha_{j+1} &= -4\alpha_j - \alpha_{j-1} \\
\beta_{j+1} &= -4\beta_j - \beta_{j-1} \\
\alpha_j &= \beta_j - 2\beta_{j-1} + \beta_{j-2}
\end{align*}
\]

Table 1 gives the values of integer constants \( \alpha_j \) and \( \beta_j \), corresponding to each integer \( j \geq 2 \). The coefficient \( d \) is given by
(2.2) \[ d = \frac{[\alpha_2(y_n - y_1) + \alpha_3(y_{n-1} - y_1) + \cdots + \alpha_n(y_2 - y_1)]}{\beta_n}. \]

In order to avoid very rapid accumulation of rounding error (which would otherwise be a serious problem if \( n \) is even moderately large), it is suggested that the division by \( \beta_n \) be postponed. Thus \( d \) would be retained in the form \( N/\beta_n \), where \( N \) is calculated exactly, using integer or fixed-point arithmetic.

The quantities \( \beta_n c_j \) (\( j = 1, 2, \ldots, n \)) are then obtained recursively by the formulas

\[
\begin{align*}
\beta_n c_1 &= \beta_n (y_2 - y_1) - N, \\
\beta_n c_j &= \beta_n (y_{j+1} - y_1) - jN - 2^3 \beta_n c_{j-1} - 3^3 \beta_n c_{j-2} - \cdots - j^3 \beta_n c_1 \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{for } j = 2, 3, \ldots, n - 1, \\
\beta_n c_n &= -\beta_n c_1 - \beta_n c_2 - \cdots - \beta_n c_{n-1},
\end{align*}
\]

again using exact calculation throughout. (The quantities \( y_j - y_1 \) must, of course, be actually multiplied by \( \beta_n \).) Finally, \( N \) and the quantities \( \beta_n c_j \) are divided by \( \beta_n \) to give the parameters \( d \) and \( c_j \) to the desired precision. It should be borne in mind that in the expression (2.1) the coefficients \( c_j \) (especially those with smaller indices) will sometimes be multiplied by large numbers, and may be needed to many decimal places.

3. Derivations and Proofs. Taking \( a = k + 1 \) in (2.1), transposing certain terms, and noting that \( s(k) = y_k \) for \( k = 1, 2, \ldots, n \) gives at once

\[ c_k = y_{k+1} - y_1 - kd - 2^3 c_{k-1} - 3^3 c_{k-2} - \cdots - k^3 c_1, \]

from which (2.4) follows immediately. Similarly, taking \( x = 2 \) gives (2.3).

Let \( \phi(x) \) denote the infinite series

\[ \phi(x) = 1^2 + 2^2 x + 3^2 x^2 + \cdots, \]

which converges in the interior of the unit circle. By actual multiplication

\[ (1 - x)^2 \phi(x) = 1 + 4x + x^2, \]

and therefore

\[ \phi(x) = \frac{1 + 4x + x^2}{(1 - x)^2}. \]

Further, let

\[ \eta(x) = \sum_{j=2}^{\infty} [s(j) - s(1)] x^{j-2}. \]

As \( s(x) \) is a linear function for \( x \geq n \), this series also converges within the unit circle, as does the binomial expansion

\[ (1 - x)^{-2} = 1 + 2x + 3x^2 + \cdots. \]

Finally, we denote by \( C(x) \) the polynomial

\[ C(x) = c_1 + c_2 x + \cdots + c_n x^{n-1}. \]
From (2.1), (3.1), (3.3), (3.4) and (3.5) we obtain the identity
\[ \eta(x) = d(1 - x)^{-2} + \phi(x)C(x). \]

Now, let
\[ \psi(x) = \frac{1}{1 + 4x + x^2}. \]

Clearly its Maclaurin expansion
\[ \psi(x) = \sum_{j=0}^{\infty} b_j x^j = 1 - 4x + 15x^2 - \cdots \]

converges in a neighborhood of the origin. Multiplying (3.6) by \((1 - x)^2 \psi(x)\) gives
\[ (1 - x)^2 \psi(x) \eta(x) = d\phi(x) + (1 - x)^{-2}C(x), \]

where we have used (3.2) and (3.7). It is shown in [1] that the coefficients \(c_j\) satisfy the two conditions
\[ c_1 + c_2 + \cdots + c_n = 0, \]
\[ c_1 + 2c_2 + \cdots + nc_n = 0. \]

Incidentally, (2.5) follows from (3.10).

Returning, however, to (3.9), we equate coefficients of \(x^{n-2}\) on both sides of that equation, noting that the coefficient of \(x^{n-2}\) in \((1 - x)^{-2}C(x)\) is
\[ (n - 1)c_1 + (n - 2)c_2 + \cdots + 2c_{n-2} + c_{n-1} = n(c_1 + c_2 + \cdots + c_n) - (c_1 + 2c_2 + \cdots + nc_n) = 0, \]

by (3.10) and (3.11). Further, let
\[ \sum_{j=0}^{\infty} a_j x^j, \]

a series having the same region of convergence as that in (3.8). We obtain, therefore,
\[ a_0(y_n - y_1) + a_1(y_{n-1} - y_1) + \cdots + a_{n-2}(y_2 - y_1) = db_{n-2}. \]

Finally, we redesignate the coefficients \(a_j\) and \(b_j\) as \(a_j\) and \(\beta_j\), shifting the indices (for notational convenience in the use of Table 1) so that \(\alpha_j = a_{j-2}\) and \(\beta_j = b_{j-2}\). Making these substitutions in (3.13) at once gives (2.2). The recurrence relation for the quantities \(\alpha_j\) follows from (3.7) and (3.12); that for the \(\beta_j\) from (3.7) and (3.8). The relation \(\alpha_j = \beta_j - 2\beta_{j-1} + \beta_{j-2}\) is an immediate consequence of (3.8) and (3.12).

4. Illustrative Example. The values of \(j\) and \(y_j\) in Table 2, due to K. A. Innanen [2], represent ten points on a segment of a theoretical rotation curve of the galactic system. Here \(y_j\) is the circular velocity in the galactic plane in km/sec at a distance of \(j\) kiloparsecs from the galactic center. Substituting in (2.2) the values of \(\alpha_j\) from Table 1 and those of \(y_j - y_1\) from Table 2 gives
\[ d = [1(-24.0) - 6(-22.5) + 24(-23.0) - \cdots + 65184(-23.0)]/40545 \]
\[ = -1005780/40545 = -67052/2703 = -24.8065. \]
Values of 2703c_j are calculated exactly, using (2.3), (2.4), and (2.5). Finally, division by 2703 gives the values of c_j, shown in the last column of Table 2 to four decimal places. Thus, the third-degree interpolating natural spline function for these data is

\[
244.0 - 24.8065(x - 1) + 1.8065(x - 1)^3 - 0.8391(x - 2)^3 \\
- 3.6437(x - 3)^3 + 2.9140(x - 4)^3 - 1.0122(x - 5)^3 \\
+ 1.1349(x - 6)^3 - 0.5272(x - 7)^3 - 0.0261(x - 8)^3 \\
+ 0.6315(x - 9)^3 - 0.4386(x - 10)^3.
\]

Mathematics Research Center, U. S. Army
University of Wisconsin
Madison, Wisconsin 53706
