Chebyshev Type Quadrature Formulas*

By David K. Kahaner

Abstract. Quadrature formulas of the form
\[ \int_{-1}^{1} f(x) \, dx \approx \frac{2}{n} \sum_{i=1}^{n} f(x_i^{(n)}) \]
are associated with the name of Chebyshev. Various constraints may be posed on the
formula to determine the nodes \( x_i^{(n)} \). Classically the formula is required to integrate \( n \)th
degree polynomials exactly. For \( n = 8 \) and \( n \geq 10 \) this leads to some complex nodes. In
this note we point out a simple way of determining the nodes so that the formula is exact
for polynomials of degree less than \( n \). For \( n = 8, 10 \) and 11 we compare our results with
others obtained by minimizing the \( l^p \)-norm of the deviations of the first \( n + 1 \) monomials
from their moments and point out an error in one of these latter calculations.

In a recent paper Barnhill, Dennis and Nielson [1] considered the possibility of
finding quadrature formulas of the form
\[ \int_{-1}^{1} f(x) \, dx \approx \frac{2}{n} \sum_{i=1}^{n} f(x_i) \]
with \( x_i \) symmetric in \([-1, 1]\) so that
\[ \sigma_n = \sum_{i=0}^{n} \left[ \frac{2}{n} \sum_{i=1}^{n} x_i^k - m_i \right]^2 \]
is minimized, where \( m_i = \int_{-1}^{1} x^i \, dx \). They have computed solutions numerically
for \( n = 8, 10 \) and 11. Classically the \( x_i \) had been determined so (1) is exact for
\( 1, x, \ldots, x^n \) but this leads to some complex nodes if \( n = 8 \) or \( n \geq 10 \).

Another possibility is to consider formulas (1) with the \( x_i \) chosen so (1) is exact
for \( 1, x, \ldots, x^p, p < n \) with \( x_i \in [-1, 1] \). This problem does not have a solution if
\( n \) and \( p \) are both required to be large [2], [3]. Nevertheless, for the small \( n \) considered
here, this problem has easily computed solutions. Although \( \sigma_n \) will not in general be
minimized, the resulting formulas have a certain appeal since polynomials of degree \( p 
\) or less can be integrated exactly.

If (1) is to be exact for \( 1, x, \ldots, x^p, p < n \), we are led to the system of equations
\[ \frac{m_k}{2} = \frac{1}{n} S_k, \quad k = 0, 1, \ldots, p, \]
where
\[ S_k = \sum_{i=1}^{n} x_i^k, \quad k = 0, 1, \ldots, p, \ldots. \]
We observe that (1') defines the sum of the first \( p \) powers of the \( n \) numbers \( x_1, \ldots, x_n \). Consequently neither \( S_{p+1}, \ldots, S_n \) nor the \( p + 1 \)st through \( n \)th symmetric function of \( x_1, \ldots, x_n \) are uniquely determined. Thus if \( \sigma_n \) and \( \sigma_{n-p} \) are \( n \)th degree polynomials whose zeros give a set of nodes for \( p = n \) and \( p < n \) respectively, then

\[
\sigma_{n-p} = \sigma_n - \pi_{n-p-1}
\]

where \( \pi_{n-p-1} \) is an arbitrary \((n - p - 1)\)st degree polynomial.** In order to characterize \( \pi \) more exactly, we use a modification of a technique in Hildebrand [4] originally due to Chebyshev. If \(|x| > \{1, |x_i|\},
\[
\int_{-1}^{1} \frac{dt}{x - t} = 2 \pi \sum_{k=1}^{n} \frac{1}{k^2} = \frac{8}{3} \pi^2.
\]

After an integration with respect to \( x \) and some manipulation we get

\[
n_{-p} \sigma_n(x) = cx^n \exp \left\{ -n \left[ \frac{1}{2 \cdot 3 x} + \frac{1}{4 \cdot 5 x^2} + \cdots \right] - \left[ \frac{b_{p+1}}{x^{p+1}} + \cdots + \frac{b_{n-1}}{x^{n-1}} + \frac{b_n}{x^n} \right] \right\}
\cdot \exp \left\{ - \sum_{k=1}^{n} \frac{b_k}{k x^k} \right\}, \quad b_k = -\frac{n}{2k} \alpha_k.
\]

Because of the left-hand side, series expansion on the right must terminate and thus neither the second exponential nor terms in first past \((1/n(n + 1))(1/x^n)\) can contribute to the polynomial part.

Thus

\[
n_{-p} \sigma_n(x) = \text{Polynomial Part} \left[ cx^n \exp \left\{ -n \sum_{j=2}^{n} c_j x^j - \sum_{j=p+1}^{n} b_j x^j \right\} \right]
\]

\[
c_i = \begin{cases} 
\frac{1}{j(j + 1)} & \text{if } j \text{ even,} \\
0 & \text{if } j \text{ odd.}
\end{cases}
\]

For \( p = n - 1 \) the contribution of \(-b_n/x^n\) appears in the constant term of \( \sigma_n \) as

\[
\sigma_n = \sigma_n - b_n = \sigma_n - S_n/n + m_n/2.
\]

For \( p = n - 2 \) the contribution of \(-b_{n-1}/x^{n-1} - b_n/x^n\) appears as

\[
\sigma_n = \sigma_n - xb_{n-1} - b_n = \frac{x n}{n-1} \left( \frac{m_{n-1}}{n} - \frac{S_{n-1}}{n} \right) - \frac{S_n}{n} + \frac{m_n}{2}.
\]

For \( n = 8 \) or \( n = 10 \) examination of the curves \( \sigma_n - b_n \) reveals that only for a small range of values of \( b_n \) does \( \sigma_n - b_n \) have \( n \) real zeros. In that case we have from (2)

\[
\sigma_n = (2b_n)^2, \quad n = 8 \text{ or } 10.
\]

** \( \pi_{n-1} \equiv 0. \)
For \( n = 11 \), \( oT_{11} \) is odd and has three real zeros. Thus \( oT_{11} - b_{11} \) cannot have all real zeros, eliminating the possibility of \( p = n - 1 \) for this case. If \( p = n - 2 = 9 \), \( oT_{11} - \alpha x - \beta \) will have 11 real zeros for appropriate \( \alpha \) and \( \beta \). From (3)

\[
\sigma_{11} = \left[ \frac{2}{11} \sum_{i=1}^{11} x_i^{10} - m_{10} \right]^2 + \left[ \frac{2}{11} \sum_{i=1}^{11} x_i^{11} - m_{11} \right]^2
\]

(4)

\[
= \left( \frac{20}{11} \alpha \right)^2 + (2\beta)^2.
\]

To preserve symmetry we set \( \beta = 0 \).

### Table I

<table>
<thead>
<tr>
<th>( n )</th>
<th>( b_n )</th>
<th>( \sigma_n(b_n) )</th>
<th>( \pm ) Nodes (( b_n ))</th>
<th>( \min \sigma_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>(-1.01117 \times 10^{-3})</td>
<td>(4.08986 \times 10^{-6})</td>
<td>0.</td>
<td>( .79221 \times 10^{-6})</td>
</tr>
<tr>
<td></td>
<td>((- .27075 \times 10^{-2} &lt; b_8 &lt; - .101117 \times 10^{-2})</td>
<td></td>
<td>(.572618)</td>
<td>(.899179)</td>
</tr>
<tr>
<td></td>
<td>for real nodes</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>(-5.955352 \times 10^{-4})</td>
<td>(1.41865 \times 10^{-6})</td>
<td>(.196220)</td>
<td>(.30362 \times 10^{-6})</td>
</tr>
<tr>
<td></td>
<td>((- .7655 \times 10^{-3} &lt; b_{10} &lt; - .5955352 \times 10^{-3})</td>
<td></td>
<td>(.196124)</td>
<td>(.920199)</td>
</tr>
<tr>
<td></td>
<td>for real nodes</td>
<td></td>
<td></td>
<td>(.645338)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \alpha )</th>
<th>( \sigma_n(\alpha) )</th>
<th>Nodes (( \alpha ))</th>
<th>( \min \sigma_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>(-3.14483 \times 10^{-4})</td>
<td>(3.26941 \times 10^{-7})</td>
<td>0.0</td>
<td>(.34535 \times 10^{-6})</td>
</tr>
<tr>
<td></td>
<td>((-8.53270 \times 10^{-4} &lt; \alpha &lt; -3.14483 \times 10^{-4})</td>
<td></td>
<td>(.264246)</td>
<td>(.264492)</td>
</tr>
<tr>
<td></td>
<td>for real nodes</td>
<td></td>
<td>(.614765)</td>
<td>(.674800)</td>
</tr>
<tr>
<td></td>
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<td></td>
<td>(.927502)</td>
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</tr>
</tbody>
</table>

In Table I we list the approximate values of \( b_8, b_{10} \) and \( \alpha \) that give all real zeros, the minimum \( \sigma_n \)'s consistent with these values and the corresponding quadrature nodes. Also we list the minimum \( \sigma_n \)'s as computed by Barnhill et al. It should be noted that their minimum \( \sigma_{11} \) is in error and the quadrature formula they have obtained corresponds roughly to selecting an \( \alpha \) at the wrong end of the allowable interval since (4) shows \( \sigma_{11} \sim \alpha^2 \). Also note that for \( n = 8 \) and \( n = 10 \) requiring (1) to be exact for (\( n - 1 \))st degree polynomials only increases \( \sigma_n \) above the computed minimum about a factor of five.

*In error. Nodes corresponding to this value of \( \sigma_n \) are \([1]; \pm .92676, \pm .70492, \pm .51792, \pm .45740, 0, 0, 0. \) Compare with nodes corresponding to \( \alpha = -8.53270 \times 10^{-4} \) and \( \sigma_n(\alpha) = 2.40684 \times 10^{-4}; \pm .925039, \pm .898403, \pm .716634, \pm .477831, \pm .477155, 0. \)
Barnhill observes that the quadrature formula corresponding to minimum $\sigma_n$ has multiple nodes at the origin. It is no longer clear if this will hold for $n = 11$. By the manner in which they were chosen our quadrature formulas should have multiple nodes at some point in $[-1, 1]$ although for $n = 10$ and 11 we have only computed these nodes so that they agree to three figures.

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