Error Bounds for Polynomial Spline Interpolation*

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Abstract. New upper and lower bounds for the $L^2$ and $L^m$ norms of derivatives of the error in polynomial spline interpolation are derived. These results improve corresponding results of Ahlberg, Nilson, and Walsh, cf. [1], and Schultz and Varga, cf. [5].

1. Introduction. In this paper, we derive new bounds for the $L^2$ and $L^m$ norms of derivatives of the error in polynomial spline interpolation. These bounds improve and generalize the known error bounds, cf. [1] and [5], in the following important ways: (1) these bounds can be explicitly calculated and are not merely asymptotic error bounds such as those given in [1] and [5]; (2) explicit lower bounds are given for the error for a class of functions; (3) the degree of regularity required of the function, $f$, being interpolated is extended, i.e., in [1] and [5] we demand that the $m$th or $2m$th derivative of $f$ be in $L^2$, if we are interpolating by splines of degree $2m - 1$, while here we demand only that some $p$th derivative of $f$, where $m \leq p \leq 2m$, be in $L^2$; and (4) bounds are given for high-order derivatives of the interpolation errors.

2. Notations. Let $-\infty < a < b < \infty$ and for each positive integer, $m$, let $K^m[a, b]$ denote the collection of all real-valued functions $u(x)$ defined on $[a, b]$ such that $u \in C^{m-1}[a, b]$ and such that $D^{m-1}u$ is absolutely continuous, with $D^mu \in L^2[a, b]$, where $Du = du/dx$ denotes the derivative of $u$. For each nonnegative integer, $M$, let $\mathcal{P}_M(a, b)$ denote the set of all partitions, $\Delta$, of $[a, b]$ of the form

(2.1) $\Delta: a = x_0 < x_1 < \cdots < x_M < x_{M+1} = b$.

Moreover, let $\mathcal{P}(a, b) = \bigcup_{M=0}^{\infty} \mathcal{P}_M(a, b)$.

If $\Delta \in \mathcal{P}_M(a, b)$, $m$ is a positive integer and $z$ is an integer such that $m - 1 \leq z \leq 2m - 2$, we define the spline space, $S(2m - 1, \Delta, z)$, to be the set of all real-valued functions $s(x) \in C^m[a, b]$ such that on each subinterval $(x_i, x_{i+1})$, $0 \leq i \leq M$, $s(x)$ is a polynomial of degree $2m - 1$. We remark that our definition is identical with the definition of deficient splines of [1]. For generalizations of this concept of spline subspace, the reader is referred to [5]. In particular, it is easy to verify that all the results of this paper remain essentially unchanged if one allows the number $z$ to depend on the partition points, $x_i$, $1 \leq i \leq M$, in such a way that $m - 1 \leq z(x_i) \leq 2m - 2$ for all $1 \leq i \leq M$. The details are left to the reader.

Following [1] we define the interpolation mapping $s_m: C^{m-1}[a, b] \to S(2m - 1, \Delta, z)$ by $s_m(f) = s$, where

(2.2) $D^ks(x_i) \equiv D^k f(x_i), \quad 0 \leq k \leq 2m - 2 - z, \quad 1 \leq i \leq M,$

$0 \leq k \leq m - 1, \quad i = 0$ and $M + 1$.

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We remark that the preceding interpolation mapping corresponds to the Type I interpolation of [1]. It is easy to modify the results of this paper for the cases in which the interpolation mapping corresponds to Types II, III, and IV interpolation of [1]. The details are left to the reader.

3. Basic $L^2$-Error Bounds. In this section, we obtain explicit upper and lower bounds for the quantities $\Lambda(m, p, z, j)$, $1 \leq m, m \leq p \leq 2m$, $m - 1 \leq z \leq 2m - 2$, and $0 \leq j \leq m$, defined by

$$\Lambda(m, p, z, j) = \sup \left\{ \frac{\| D^j f - \varphi_m f \|^2_{L^2(a, b)}}{\| D^p f \|^2_{L^2(a, b)}} : f \in K^p[a, b], \frac{\| D^j f \|^2_{L^2(a, b)}}{\| D^p f \|^2_{L^2(a, b)}} \neq 0 \right\}.$$  

(3.1)

First, we recall some basic results from [1] and [5] and introduce some additional notation.

**Theorem 3.1.** The interpolation mapping given by (2.2) is well defined for all $\Delta \in \Theta(a, b)$, $1 \leq m$, and $m - 1 \leq z \leq 2m - 2$.

**Theorem 3.2 (First Integral Relation).** If $f \in K^m[a, b]$, $1 \leq m$, $\Delta \in \Theta(a, b)$, and $m - 1 \leq z \leq 2m - 2$,

$$\| D^m f \|^2_{L^2(a, b)} = \| D^m f - \varphi_m f \|^2_{L^2(a, b)} + \| D^m \varphi_m f \|^2_{L^2(a, b)}.$$  

(3.2)

**Theorem 3.3 (Second Integral Relation).** If $f \in K^{2m}[a, b]$, $1 \leq m$, $\Delta \in \Theta(a, b)$, and $m - 1 \leq z \leq 2m - 2$,

$$\| D^m f - \varphi_m f \|^2_{L^2(a, b)} = \int_a^b (f - \varphi_m f) D^{2m} f \, dx.$$  

(3.3)

Finally, following Kolmogorov, cf. [4, p. 146], if $t$ and $d$ are positive integers, let $\lambda_d(t)$ denote the $d$th eigenvalue of the boundary value problem

$$(-1)^t D^2 y(x) = \lambda y(x), \quad a < x < b,$$  

(3.4)

$$D^k y(a) = D^k y(b) = 0, \quad t \leq k \leq 2t - 1,$$  

(3.5)

where the $\lambda_d$ are arranged in order of increasing magnitude and repeated according to their multiplicity. We remark that the problem (3.4)–(3.5) has a countably infinite number of eigenvalues, all of which are nonnegative and it may be shown that

$$\lambda_d = (\pi/(b - a))^{2t} d^{2t}[1 + O(d^{-1})], \quad \text{as } t < d \to \infty.$$  

Using the bootstrapping technique of [1, p. 92], and letting

$$\bar{\Delta} \equiv \max_{0 \leq t \leq M} (x_{t+1} - x_t) \quad \text{and} \quad \underline{\Delta} \equiv \min_{0 \leq t \leq M} (x_{t+1} - x_t),$$

for all $\Delta \in \Theta_M(a, b)$, we have the following generalization of Theorem 7 of [5].

**Theorem 3.4.**

$$\lambda_d^{-1/2}(m - j) \leq \Lambda(m, m, z, j) \leq K_{m, m, z, j}(\bar{\Delta})^{m - j},$$  

(3.6)

where

$$d = (M + 1)(2m - z + 1) + z - j + 2.$$  

(3.7)
and

\begin{align}
K_{m,m,z,j} &= 1, &\text{if } m - 1 \leq z \leq 2m - 2, j = m, \\
&= (1/\pi)^{m-j}, &\text{if } m - 1 = z, 0 \leq j \leq m - 1, \\
&= (z + 2 - m)! / \pi^{m-j}, &\text{if } m - 1 \leq z \leq 2m - 2, 0 \leq j \leq 2m - 2 - z, \\
&= (z + 2 - m)! / j! \pi^{m-j}, &\text{if } m - 1 \leq z \leq 2m - 2, 2m - 2 - z \leq j \leq m - 1,
\end{align}

for all $1 \leq m, 0 \leq M, \Delta \subseteq \mathcal{P}_{m\Delta}(a, b), m - 1 \leq z \leq 2m - 2,$ and $0 \leq j \leq m.$

**Proof.** First, we prove the right-hand inequality of (3.6). If $m - 1 \leq z \leq 2m - 2$ and $j = m,$ the result follows directly from Theorem 3.2.

Otherwise, $D^i(f - s_m f)(x_i) = 0, 1 \leq i \leq M, 0 \leq j \leq 2m - 2 - z,$ and by the Rayleigh-Ritz inequality, cf. [3, p. 184],

\begin{align}
\int_{x_i}^{x_{i+1}} (D^i(f - s_m f)(x))^2 \, dx &\leq \left( \frac{\Delta}{\pi} \right)^2 \int_{x_i}^{x_{i+1}} (D^{i+1}(f - s_m f)(x))^2 \, dx,
\end{align}

$0 \leq j \leq 2m - 2 - z.$ Summing both sides of (3.9) with respect to $i$ from 0 to $M,$ we obtain

\begin{align}
\|D^i(f - s_m f)\|_{L^2[a, b]} &\leq \left( \frac{\Delta}{\pi} \right)^2 \|D^{i+1}(f - s_m f)\|_{L^2[a, b]},
\end{align}

$0 \leq j \leq 2m - 2 - z.$ Using (3.10) repeatedly we obtain

\begin{align}
\|D^i(f - s_m f)\|_{L^2[a, b]} &\leq \left( \frac{\Delta}{\pi} \right)^{2m-1-z-i} \|D^{2m-1-z}(f - s_m f)\|_{L^2[a, b]}.
\end{align}

Hence, if $2m - 1 - z = m,$ i.e., $z = m - 1,$ then

\begin{align}
\|D^i(f - s_m f)\|_{L^2[a, b]} &\leq \left( \frac{1}{\pi} \right)^{m-i} (\Delta)^{m-i} \|D^m(f - s_m f)\|_{L^2[a, b]},
\end{align}

which is the required result for this special case.

Otherwise, since $m \leq z,$ applying Rolle's Theorem to $D^{2m-2-z}(f - s_m f) \in C_{m-2}^\infty[a, b],$ which vanishes at every mesh point, we have that for each $0 \leq j \leq z - m + 1,$ there exist points $\{\xi^{(i)}_l\}_{l=0}^{M+1}$ in $[a, b]$ such that

\begin{align}
D^{2m-2-z}(f - s_m f)(\xi^{(i)}_l) &= 0, &0 \leq j \leq m - 1 - (2m - 2 - z), \\
&= z - m + 1, &0 \leq l \leq M + 1 - j,
\end{align}

\begin{align}
a = \xi^{(i)}_0 < \xi^{(i)}_1 < \cdots < \xi^{(i)}_{z-1} = b, &0 \leq j \leq z - m + 1,
\end{align}

\begin{align}
\xi^{(i)}_l \leq \xi^{(i+1)}_l < \xi^{(i)}_{l+1}, &\text{ for all } 0 \leq l \leq M + 1 - j, 0 \leq j \leq z - m + 1
\end{align}

and

\begin{align}
|\xi^{(i)}_{l+1} - \xi^{(i)}_l| &\leq (j + 1)\Delta, &0 \leq l \leq M - j, 0 \leq j \leq z - m + 1,
\end{align}

i.e., choose $\xi^{(i)}_0 = x_i, 0 \leq l \leq M + 1.$
Thus, applying the Rayleigh-Ritz inequality, we have

$$\int_{t_{i+1}}^{t_i} \left( D^{2n-2-s+i}(f - \vartheta_m)(x) \right)^2 \, dx$$

(3.17)

$$\leq \left[ \frac{(j + 1)A}{\pi} \right]^2 \int_{t_{i+1}}^{t_i} \left( D^{2n-2-s+i+1}(f - \vartheta_m) \right)^2 \, dx$$

for all $0 \leq l \leq M - j$, $0 \leq j \leq z - m + 1$. Summing (3.17) with respect to $l$ from 0 to $M - j$, we have

$$\left\| D^{2n-2-s+i}(f - \vartheta_m) \right\|_{L^2[a, b]} \leq \frac{(j + 1)A}{\pi} \left\| D^{2n-2-s+i+1}(f - \vartheta_m) \right\|_{L^2[a, b]},$$

(3.18) for all $0 \leq j \leq z - m + 1$. Using (3.18) repeatedly along with (3.2) we have

$$\left\| D^{2n-1-s+i}(f - \vartheta_m) \right\|_{L^2[a, b]} \leq \frac{(j + 2 - m)!}{\pi^{s-m+1}} \left( \Delta \right)^{s-m+1} \left\| D^{n}(f - \vartheta_m) \right\|_{L^2[a, b]},$$

(3.19)

Combining (3.11) with (3.19), we have that

$$\left\| D^t(f - \vartheta_m) \right\|_{L^2[a, b]} \leq \frac{(2m - 2 - z)!}{\pi^{s}} \left( \Delta \right)^{s} \left\| D^{n}(f - \vartheta_m) \right\|_{L^2[a, b]},$$

(3.20) if $0 \leq j \leq 2m - 2 - z$. Otherwise, it follows from (3.18) that

$$\left\| D^t(f - \vartheta_m) \right\|_{L^2[a, b]} \leq \frac{(2m - 2 - z)!}{j!} \left( \Delta \right)^{s-j+1} \left\| D^{n}(f - \vartheta_m) \right\|_{L^2[a, b]},$$

(3.21)

Finally, we prove the left-hand inequality of (3.6). This inequality follows directly from a fundamental result of Kolmogorov, cf. [4, p. 146], which states that

$$\lambda_{t+3/2}(m - j) \leq \Lambda(m, m, z, j),$$

(3.22)

where $t = \text{dimension } D^t(S(2m - 1, \Delta, z))$, for all $1 \leq m$, $0 \leq M$, $\Delta \in \mathcal{O}_m(a, b)$, $m - 1 \leq z \leq 2m - 2$, and $0 \leq j \leq m$. But the space $D^t(S(2m - 1, \Delta, z))$ has dimension $t = (2m - j)(M+1) - (z+1-j)M = (M+1)(2m-z+1) + z-j+1$.

Q.E.D.

We remark that in this case it is easy to verify that there exists a positive constant, $K$, such that

$$\lambda_{2-t/2} \geq \left( \frac{b - a}{\pi} \right)^{n-t} \frac{1}{(M + 1)^{n-t}} \frac{1}{s^{n-t}} \frac{1}{1 + Ks^{-1}(M + 1)^{-1}}$$

$$\geq \frac{1}{\pi^{n-t}} \frac{1}{s^{n-t}} \frac{1}{1 + Ks^{-1}(M + 1)^{-1}} \left( \Delta \right)^{n-t},$$

where $s = (2m - z + 1 + (z - j + 2)/(M + 1))$, and thus that splines are "quasi-optimal".

The next result generalizes Theorem 9 of [5].

**Theorem 3.5.**

$$\lambda_{2-t/2} \leq \Lambda(m, 2m, z, j) \leq K_m, m, z, j \leq \Lambda(m, 2m, \ldots, (\Delta)^{s-m-t})$$

(3.23)
where

\[(3.24)\quad d \equiv (M + 1)(2m - z + 1) + z - j + 2\]

and

\[(3.25)\quad K_{m,2m,z,j} \equiv (K_{m,m,z,j})(K_{m,m,z,0}), \quad \text{for all } 1 \leq m, 0 \leq M, \Delta \in \mathcal{O}_M(a, b),\]

\[m - 1 \leq z \leq 2m - 2, \text{ and } 0 \leq j \leq m.\]

**Proof.** Applying the Cauchy-Schwarz inequality to the Second Integral Relation yields the inequality

\[(3.26)\quad ||D^m(f - \mathcal{S}_m f)||_{L^2[a,b]} \leq ||D^{2m}|| ||L^2[a,b]|| ||f - \mathcal{S}_m f||_{L^2[a,b]}.

Applying the proof of Theorem 3.4, we have

\[(3.27)\quad ||D^j(f - \mathcal{S}_m f)||_{L^2[a,b]} \leq K_{m,m,z,j} ||D^{2m}(f - \mathcal{S}_m f)||_{L^2[a,b]} \Delta^{m-j}.

Using (3.27) for the special case of \(j = 0\) in (3.26) yields

\[(3.28)\quad ||D^m(f - \mathcal{S}_m f)||_{L^2[a,b]} \leq ||D^{2m}|| ||L^2[a,b]|| ||\mathcal{S}_m f||_{L^2[a,b]} \Delta^m.

Using (3.28) to bound the right-hand side of (3.27) gives us the right-hand inequality of (3.23). The left-hand inequality of (3.23) follows as in Theorem 3.4. Q.E.D.

We now recall a fundamental inequality of E. Schmidt which will be used several times in the remainder of this paper.

**Lemma 3.1.** If \(p_N(x)\) is a polynomial of degree \(N\),

\[(3.29)\quad ||D^p p_N||_{L^2[a,b]} \leq \frac{E_N}{b-a} ||p_N||_{L^2[a,b]},\]

where \(E_N = (N + 1)^2 \sqrt{2}\).

**Proof.** Cf. [2]. Q.E.D.

**Theorem 3.6.**

\[(3.30)\quad \lambda_d^{-1/2}(p - j) \leq \Delta(m, p, z, j) \leq K_{m,p,z,j} \Delta^{p-j},

where

\[(3.31)\quad d \equiv (M + 1)(2m - z + 1) + z - j + 2\]

and

\[(3.32)\quad K_{m,p,z,j} \equiv \left\{ K_{p,p,2m-1,j} + K_{m,2m,z,j} \cdot 2^{(1/2)(2m-p)} \left[ \frac{p!}{(2p - 2m)!} \right]^2 \frac{(\Delta/\Delta)^{2m-p}}{\Delta}\right\}

for all \(1 \leq m, 0 \leq M, \Delta \in \mathcal{O}_M(a, b), m < p < 2m, 4m - 2p - 1 \leq z \leq 2m - 2, \text{ and } 0 \leq j \leq m.\)

**Proof.** Consider \(S(2p - 1, \Delta, 2m - 1) \subset K^{2m}[a, b]\). This space is well defined since \(2p - 2 \geq 2(m + 1) - 2 = 2m\). Moreover, if \(\mathcal{S}_m\) denotes the interpolation mapping of \(C^{m-1}[a, b]\) into \(S(2m - 1, \Delta, z)\) and \(\mathcal{S}_p\) denotes the interpolation mapping of \(C^{p-1}[a, b]\) into \(S(2p - 1, \Delta, 2m - 1)\), then \(\mathcal{S}_m(\mathcal{S}_p f) = \mathcal{S}_m f\) for all \(f \in C^{p-1}[a, b]\). In fact, \(D^i \mathcal{S}_p f\) interpolates \(D^j f\) at \(x_i\), \(1 \leq i \leq M\), for all \(0 \leq k \leq 2p - (2m - 1) - 2 = 2p - 2m - 1\), while \(D^i \mathcal{S}_m f\) interpolates \(D^j f\) at \(x_i\), \(1 \leq i \leq M\), for all \(0 \leq k \leq 2m - z - 2 \leq 2m - (4m - 2p - 1) - 2 = 2p - 2m - 1\).
Thus,

\[ ||D^j(f - s_m f)||_{L^1[a, b]} \leq ||D^j(f - s_p f)||_{L^1[a, b]} \]

\[ + ||D^j(s_p f - s_m(s_p f))||_{L^1[a, b]}, \quad 0 \leq j \leq m. \]

By Theorem 3.4,

\[ ||D^j(f - s_p f)||_{L^1[a, b]} \leq K_{p, p, 2m-1, j} \langle \Delta \rangle^{r-j} \quad (3.34) \]

and by Theorem 3.5

\[ ||D^j(s_p f - s_m(s_p f))||_{L^1[a, b]} \leq K_{m, 2m, z, j} \langle \Delta \rangle^{2m-j} \quad (3.35) \]

But by Schmidt's inequality and the First Integral Relation, since \( s_p f \) is a piecewise polynomial of degree \( 2p - 1 \) with \( p > m \), we have

\[ ||D^j(s_p f||_{L^1[a, b]} \leq \frac{\left( \sum_{l=1}^{2m-p} E_{2p-2m-1+l} \right) ||D^j f||_{L^1[a, b]}}{(\Delta)^{2m-p}} \]

\[ \leq 2^{(2m-p)/2} \left( \frac{p!}{(2p + 2m)!} \right)^2 \quad (3.36) \]

The required result now follows from (3.33), (3.34), (3.35), and (3.36). Q.E.D.

4. \( L^2 \)-Error Bounds for Higher Order Derivatives. In this section we give explicit upper bounds for the quantities \( \Lambda (m, p, z, j) \) in the special cases of \( m < p - 2m \) and \( m < j - p \). Since \( s_p f \) is not necessarily in \( K^r[a, b] \) if \( z + 1 < j \leq p \), it is necessary to modify the definition of \( \Lambda (m, p, z, j) \) given in (3.1). The new definition is given by

\[ \Lambda (m, p, z, j) = \sup \left\{ \left( \sum_{i=0}^{M} ||D^i(f - s_m f)||_{L^1[a, b]} \right)^{1/2} \right\} \quad (4.1) \]

The main result of this section is

**Theorem 4.1.**

\[ ||D^j(f - s_p f)||_{L^1[a, b]} \leq K_{m, p, z, j} \langle \Delta \rangle^{p-j} \quad (4.2) \]

where

\[ K_{m, p, z, j} = \left[ K_{p, p, z} + (K_{m, z, m} + K_{p, p, m})2^{(j-m)/2} \left( \frac{(2p + m)!}{(2p - j)!} \right) \langle \Delta \rangle^{j-m} \right] \]

for all \( 1 \leq m, 0 \leq M, \Delta \in \Phi_m(a, b), m < p \leq 2m, 4m - 2p - 1 \leq z \leq 2m - 2, \) and \( m < j \leq p \).

**Proof.** By Theorem 3.6,

\[ ||D^m(f - s_m f)||_{L^1[a, b]} \leq K_{m, p, z, m} \langle \Delta \rangle^{p-m} \quad (4.3) \]

and by Theorem 3.4,

\[ ||D^k(f - s_p f)||_{L^1[a, b]} \leq K_{p, p, p, h} \langle \Delta \rangle^{k-p} \quad (4.4) \]

for all \( 1 \leq m, 0 \leq M, \Delta \in \Phi_m(a, b), m < p \leq 2m, 4m - 2p - 1 \leq z \leq 2m - 2, \) and \( m < j \leq p \).
Combining (4.4) and (4.5), we obtain

\[ ||D^n(s_m f - s_p f)||_{L^r[a, b]} \leq (K_{m, p, z, m} + K_{p, p, p, m})(\Delta)^{p-m}. \]

Using the Schmidt inequality in (4.6), we obtain

\[ ||D^i(s_m f - s_p f)||_{L^r[a, b]} \leq \frac{\left( \prod_{i=1}^{m} E_{(2p-1)-j+r} \right)}{(\Delta)^{p-i}} ||D^p(s_m f - s_p f)||_{L^r[a, b]} \]

\[ \leq (K_{m, p, z, m} + K_{p, p, p, m}) \left( \prod_{i=1}^{m} E_{2p-1-j+r} \right) (\Delta)^{p-i}(\Delta/\Delta)^{i-m}. \]

The required result follows from (4.5), (4.7), and

\[ ||D^i(f - s_m f)||_{L^r[a, b]} \leq ||D^i(f - s_p f)||_{L^r[a, b]} + ||D^i(s_p f - s_m f)||_{L^r[a, b]}. \]

Q.E.D.

We remark that in those cases in which \( s_m f \in K'[a, b] \), lower bounds of the form introduced in Section 3 can be given for \( \Lambda(m, p, z, j) \).

5. \( L^p \)-Error Bounds. In this section, we give explicit upper bounds for the quantities \( \Lambda^p(m, p, z, j) \), \( 1 \leq m, m \leq p \leq 2m, m - 1 \leq z \leq 2m - 2, \) and \( 0 \leq j \leq p \), defined by

\[ \Lambda^p(m, p, z, j) = \sup \left\{ \max_{0 \leq \xi \leq M} \left( \frac{||D^i(f - s_m f)||_{L^r[a, b]} ||D^j f||_{L^r[a, b]} }{||D^i f||_{L^r[a, b]} } \right) : f \in K^n[a, b], ||D^j f||_{L^r[a, b]} \neq 0 \right\}. \]

We obtain the following results as corollaries of the results of Section 3 and Section 4. As an improvement of Theorem 6 of [5], we have

**Theorem 5.1.**

\[ \Lambda^p(m, m, z, j) \leq K_{m, m, z, j}(\Delta)^{m-j-1/2}, \]

where

\[ K_{m, m, z, j} = \begin{cases} K_{m, m, z, j+1} & \text{if } m - 1 = z, 0 \leq j \leq m - 1, \\ K_{m, m, z, j+1} & \text{if } m - 1 < z \leq 2m - 2, 0 \leq j \leq 2m - 2 - z, \\ (j - 2m + 3 + z)^{1/2} K_{m, m, z, j+1} & \text{if } m - 1 < z \leq 2m - 2, \\ 2m - 2 - z < j \leq m - 1, \end{cases} \]

for all \( 1 \leq m, 0 \leq M, \Delta \in \mathcal{I}(a, b), m - 1 \leq z \leq 2m - 2, \) and \( 0 \leq j \leq m - 1 \).

**Proof.** We give the proof in the special case of \( m - 1 = z, 0 \leq j \leq m - 1, \) as the proof in the other cases is analogous. Given any \( x \in [a, b] \), there exists a point \( y \in [a, b] \) such that \( D^i(f - s_m f)(y) = 0 \) and \( |x - y| \leq \Delta \). Hence, \( D^i(f - s_m f)(x) = \int_x^y D^{i+1}(f - s_m f)(t) \, dt \) and

\[ ||D^i(f - s_m f)||_{L^r[a, b]} \leq (\Delta)^{1/2} ||D^{i+1}(f - s_m f)||_{L^r[a, b]}. \]

The result now follows from applying Theorem 3.4 to the right-hand side of the preceding inequality. Q.E.D.
As in Theorem 5.1, we have as an improvement of Theorem 8 of [5].

Theorem 5.2.

\[ \Lambda^\omega(m, 2m, z, j) \leq K^\omega_{m, 2m, z, j}(\Delta)^{2m-j-1/2}, \]

where

\[ K^\omega_{m, 2m, z, j+1} = K_{m, 2m, z, j+1}, \quad \text{if } m - 1 = z, 0 < j \leq m - 1, \]
\[ = K_{m, 2m, z, j+1}, \quad \text{if } m - 1 < z \leq 2m - 2, 0 \leq j \leq 2m - 2 - z, \]
\[ = (j - 2m + 3 + z)^{1/2} K_{m, 2m, z, j+1}, \quad \text{if } m - 1 < z \leq 2m - 2, \]
\[ = 0 - 2m + 3 + z)^{1/2} K_{m, 2m, z, j+1}, \quad \text{if } m - 1 < z \leq 2m - 2, \]
\[ \text{for all } 1 < M, \Delta \in \Theta_m(a, b), m - 1 \leq z \leq 2m - 2, \text{ and } 0 \leq j \leq m - 1. \]

As in Theorem 3.6, we have

Theorem 5.3.

\[ \Lambda^\omega(m, p, z, j) \leq K^\omega_{m, p, z, j}(\Delta)^{p-j-1/2}, \]

where

\[ K^\omega_{m, p, z, i} = \left\{ \frac{p!}{(2p - 2m)!} \right\} \left( \frac{\Delta}{\Delta} \right)^{2m-p} \right\}, \]

for all \( 1 \leq m, 0 \leq M, \Delta \in \Theta_m(a, b), m < p < 2m, 4m - 2p - 1 \geq z \geq 2m - 2, \)
\[ \text{and } 0 \leq j \leq m - 1. \]

Finally, to give a result analogous to Theorem 4.1, we need an inequality due to A. A. Markov.

Lemma 5.1. If \( p_N(x) \) is a polynomial of degree \( N \), then

\[ ||D^q p_N||_{L^a,b} \leq M_N \frac{M_N}{b - a} ||p_N||_{L^a,b}, \]

where \( M_N = 2N^2 \).

Proof. Cf. [6]. Q.E.D.

As an extension of Theorem 10 of [5], we prove

Theorem 5.4.

\[ \Lambda^\omega(m, p, z, j) \leq K^\omega_{m, p, z, j}(\Delta)^{p-j-1/2}, \]

where

\[ K^\omega_{m, p, z, i} = \left\{ \frac{(2p - m)!}{(2p - j - 1)!} \right\} \left( \frac{\Delta}{\Delta} \right)^{i-m+1} \right\}, \]

for all \( 1 \leq m, 0 \leq M, \Delta \in \Theta_m(a, b), m < p \leq 2m, 4m - 2p - 1 \leq z \leq 2m - 2, \)
\[ \text{and } m - j - p - 1. \]

Proof. From Theorem 5.1, we have that

\[ ||D^k(f - \sigma_m)||_{L^a,b} \leq K_{m, p, z, i}(\Delta)^{p-k-1/2} ||D^k||_{L^a,b}, \quad \text{for } 0 \leq k \leq p - 1, \]

and from Theorem 5.3

\[ ||D^{m-1}(f - \sigma_m)||_{L^a,b} \leq K_{m, p, z, m-1}(\Delta)^{p-m+1/2} ||D^m||_{L^a,b}. \]
Combining (5.11) and (5.12), we have

\begin{equation}
\|D^{m+1}(s_m - s_p)\|_{L^m[a,b]} \leq (K_{m,p}^{m,1} + K_{p,p}^{m,1}) (\Delta)^{p-m+1/2} \|D^p f\|_{L^p[a,b]}.
\end{equation}

But,

\begin{equation}
\|D^j(s_m - s_p)\|_{L^m[a,b]} \leq \frac{\left(\prod_{i=1}^{j-m+1} M_{2p-1-f+i}(\Delta)^{j-m+1}\right)}{(2p-j-1)!} \frac{1}{(\Delta)^{j-m+1}} \|D^{m+1}(s_m - s_p)\|_{L^m[a,b]}.
\end{equation}

The required result follows directly from (5.11), (5.13), (5.14), and the observation that

\begin{equation}
\|D^j f\|_{L^m[a,b]} \leq \|D^j f - s_m\|_{L^m[a,b]} + \|D^j(s_m - s_p)\|_{L^m[a,b]}.
\end{equation}

Q.E.D.

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