Remark on a Conjecture of Erdős on Binomial Coefficients

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Abstract. A conjecture attributed to Erdős concerning the Diophantine equation

\[ 2 \binom{x + n - 1}{n} = \binom{y + n - 1}{n} \]

is shown to be false.

M. Wunderlich [2] attributes the following conjecture to P. Erdős:

The equation

\[ 2 \binom{x + n - 1}{n} = \binom{y + n - 1}{n} \]

has only one solution in positive integers: \( x = n, y = n + 1 \).

Because (1) has infinitely many solutions for \( n = 2 \) (cf. [1, p. 30]) the assumption \( n \geq 3 \) must surely be added. But that does not suffice.

Observe that for \( b - a \geq 3 \) the equality

\[ s \binom{a}{2} = t \binom{b}{2} \]

implies

\[ s \binom{b - 2}{b - a} = t \binom{b}{b - a} \].

Because (2) has infinitely many solutions in integers \( a, b \) for \( s = 2, t = 1 \), we obtain infinitely many counterexamples to the conjecture of Erdős, viz. \( n = b - a, x = a - 1, y = a + 1 \), where

\[ 2 \binom{a}{2} = \binom{b}{2} \]

For example,

\[ 2 \binom{19}{6} = \binom{21}{6} \]

is a solution of (1).

Probably the conjecture is true when we require \( y - x \geq 3 \).

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