

# Error Estimates for a Chebyshev Quadrature Method

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**Abstract.** Filippi [1] has proposed a quadrature scheme for any function  $f(x)$  in  $[-1, 1]$ , based on expanding the integrand in a series of Chebyshev polynomials of the second kind. In this paper the error associated with this quadrature method when applied to analytic functions has been investigated in detail.

**Introduction.** A Chebyshev polynomial of the second kind is  $U_{n-1}(x) \equiv (1/n)T'_n(x)$  where  $T_n(x)$  is the Chebyshev polynomial of the first kind of degree  $n$ , defined by  $T_n(x) = \cos n\theta$  with  $x = \cos \theta$ . Accordingly, we shall follow Filippi and consider the expansion of function in terms of the  $T'_n(x)$  instead of  $U_n(x)$ .

Let  $f(x)$ , a function of bounded variation in  $[-1, 1]$ , be expanded in a series of  $T'_n(x)$  as

$$(1) \quad f(x) = \sum_{n=1}^{\infty} a_n T'_n(x),$$

where

$$(2) \quad a_n = \frac{2}{\pi n^2} \int_{-1}^1 (1-x^2)^{1/2} T'_n(x) f(x) dx.$$

In general, the integral in (2) cannot be evaluated explicitly and recourse has to be made to approximate methods for evaluating  $a_n$  and then to obtain a suitable polynomial approximation to  $f(x)$ .

Filippi [1], has approximated the function  $f(x)$  by a polynomial  $\psi_{N-1}(x)$  of degree  $N - 1$ , by collocation with  $f(x)$  at the  $N$ -points, which are the zeros of  $T'_{N+1}(x)$  and has obtained a quadrature formula for  $f(x)$  by integrating  $\psi_{N-1}(x)$ .

In the first section of this paper, the contour integral estimate of error  $\bar{\psi}_{N-1}(x) = f(x) - \psi_{N-1}(x)$  is considered in brief and in the subsequent sections the error in the Filippi quadrature scheme for analytic functions is discussed. Analogous investigation on Clenshaw-Curtis quadrature [2] and Gaussian quadrature have been made by Chawla [3], [4] and Chawla and Jain [5].

**1. Contour Integral Estimate of  $\bar{\psi}_{N-1}(x)$ .** Following Filippi [1], let  $f(x)$  be approximated by a polynomial  $\psi_{N-1}(x)$  of degree  $N - 1$  over the zeros of  $T'_{N+1}(x)$ , so that

$$(3) \quad \psi_{N-1}(x) = \sum_{n=1}^N B_{n,N} T'_n(x).$$

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The coefficients  $B_{n,N}$  are calculated by trigonometric interpolation as in [1].

$$(4) \quad B_{n,N} = \frac{2}{n^2(N+1)} \sum_{i=1}^N (1-x_i^2) T'_n(x_i) f(x_i),$$

where

$$(5) \quad x_i = \cos \frac{\pi i}{N+1}, \quad i = 1(1)N.$$

To obtain the contour integral estimate of  $\bar{\psi}_{N-1}(x)$  we consider the function  $f(z)$ , where  $z = x + iy$ . By Cauchy's integral formula we can represent  $f(x)$  by

$$(6) \quad f(x) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-x} dz,$$

where  $C$  is any contour on and within which  $f(z)$  is regular. If the contour  $C$  is so chosen that it contains the interval  $-1 \leq x \leq 1$ , then selecting the abscissas as the zeros of  $T'_{N+1}(x)$ , the Lagrange interpolation polynomial for  $f(x)$  can be written using [6, Section 3.6, p. 67] as

$$(7) \quad \psi_{N-1}(x) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-x)T'_{N+1}(z)} [T'_{N+1}(z) - T'_{N+1}(x)] dz,$$

where the error

$$(8) \quad \bar{\psi}_{N-1}(x) = \frac{T'_{N+1}(x)}{2\pi i} \int_C \frac{f(z)}{(z-x)T'_{N+1}(z)} dz.$$

**2. Error in the Filippi Quadrature Method.** In the Filippi quadrature formula we have from (3)

$$(9) \quad \int_{-1}^1 f(x) dx \approx \int_{-1}^1 \psi_{N-1}(x) dx = 2 \sum_{i=0}^M B_{2i+1,N},$$

where  $M = (N-1)/2$ , if  $N$  is odd and  $M = (N-2)/2$  for even  $N$ . Substituting for  $B_{2i+1,N}$ , the expression in (4), (9) becomes

$$(10) \quad \int_{-1}^1 f(x) dx \approx \sum_{i=1}^N \lambda_i f(x_i),$$

where

$$(11) \quad \lambda_i = \frac{4}{N+1} \sum_{j=0}^M \frac{(1-x_i^2)}{(2j+1)^2} T'_{2i+1}(x_i).$$

The error in the quadrature formula follows from (8) as

$$(12) \quad E_{N-1}(\psi) = \int_{-1}^1 \bar{\psi}_{N-1}(x) dx = \frac{1}{\pi i} \int_C \frac{f(z)L_N(z)}{T'_{N+1}(z)} dz,$$

where we have put

$$(13) \quad L_N(z) = \frac{1}{2} \int_{-1}^1 \frac{T'_{N+1}(x)}{z-x} dx.$$

Equation (13) defines  $L_N(z)$  as a single-valued analytic function in the  $z$ -plane with

the interval  $[-1, 1]$  for  $x$  deleted. In the following discussion we shall work out an error estimate of the quadrature formula but before that we prove a lemma for  $L_N(z)$  for odd  $N$ .

**3. Lemma for  $L_N(z)$ .** Let us now introduce the mapping

$$(14) \quad z = \frac{1}{2}(\xi + \xi^{-1}), \quad \xi = \rho e^{i\theta}, \quad 0 \leq \theta \leq 2\pi.$$

This maps the exterior of the unit circle  $|\xi| = 1$  conformally onto the  $z$ -plane with the interval  $-1 \leq x \leq 1$  deleted. The circle  $|\xi| = \rho, \rho > 1$ , maps onto the ellipse  $\mathcal{E}_\rho$  with foci at  $z = \pm 1$  and semiaxes  $\frac{1}{2}(\rho + \rho^{-1})$  and  $\frac{1}{2}(\rho - \rho^{-1})$ . The lemma stated below gives a simple representation of  $L_N(z)$  on  $\mathcal{E}_\rho$ .

LEMMA. For  $z \in \mathcal{E}_\rho$

$$(15) \quad |L_N(z)| \leq (N + 1) \left[ \frac{\sigma_{N,N+3}}{\rho^2 - 1} + \frac{1}{\rho^{N+1}} \frac{4(N + 1)}{2N + 3} \right] \quad \text{for odd } N,$$

where

$$(16) \quad \sigma_{N,N+3} = 2 \sum_{m=1}^{N-1} \frac{1}{2m + 1}.$$

*Proof.* Following Davis [6, Lemma 12.4.6, p. 311] we set  $x = \cos \theta$  and transform (13) to the  $\xi$  plane. Then,

$$(17) \quad L_N(z) = (N + 1) \int_0^\pi \frac{\xi^{-1}}{1 - 2\xi^{-1} \cos \theta + \xi^{-2}} \sin(N + 1)\theta \, d\theta.$$

Now

$$(18) \quad \frac{\xi^{-1}}{1 - 2\xi^{-1} \cos \theta + \xi^{-2}} = \sum_{p=1}^\infty \xi^{-p} \frac{\sin p\theta}{\sin \theta}.$$

The series converges uniformly and absolutely for  $0 \leq \theta \leq \pi$  and for all  $|\xi| \geq \rho > 1$

Substituting (18) in (17),

$$(19) \quad L_N(z) = (N + 1) \sum_{p=1}^\infty \sigma_{N,p} \xi^{-p},$$

where

$$(20) \quad \sigma_{N,p} = \int_0^\pi \frac{\sin(N + 1)\theta \sin p\theta}{\sin \theta} \, d\theta.$$

From (20) we get

$$(21) \quad \begin{aligned} \sigma_{N,p} &= 0 \quad \text{if } N - p \text{ is even,} \\ &= 2 \sum_{m=1}^p \frac{1}{N - p + 2m}, \quad \text{if } N - p \text{ is odd.} \end{aligned}$$

It can now be easily verified that  $\sum_{m=1}^p 1/(N - p + 2m)$  assumes its maximum value when  $N - p = -1$  for a fixed  $N$ . In that case  $\sigma_{N,N+1}$  is the greatest coefficient in (19) and the next highest coefficient is  $\sigma_{N,N+3}$ , where

$$(22) \quad \sigma_{N,N+1} = \sigma_{N,N+3} + \frac{4(N + 1)}{2N + 3}.$$

Now since  $N$  is odd, by taking only even values of  $p$  (say  $2r$ ), we have, from (19) and (22)

$$(23) \quad L_N(z) < (N + 1) \left[ \sigma_{N,N+3} \sum_{r=1}^{\infty} \xi^{-2r} + \frac{4(N + 1)}{2N + 3} \xi^{-(N+1)} \right]$$

and as  $|\xi| = \rho > 1$ , the lemma is easily established.

**4. Error Estimate for Analytic Functions.** To obtain the error estimate for analytic functions in  $[-1, 1]$ , a suitable choice of the contour in (12) is an ellipse as defined in (14). Now if  $f \in A[-1, 1]$ , then for some  $\rho > 1$ ,  $f$  can be continued analytically so as to be regular in the closed ellipse  $\mathcal{E}_\rho$ . Also on  $\mathcal{E}_\rho$  both  $T'_{N+1}(z)$  and  $L_N(z)$  have simple representations. That is, on  $\mathcal{E}_\rho$

$$(24) \quad T'_{N+1}(z) = (N + 1) \left\{ \frac{\xi^{N+1} - \xi^{-(N+1)}}{\xi - \xi^{-1}} \right\}$$

and

$$(25) \quad \frac{|dz|}{|T'_{N+1}(z)|} \leq \frac{1}{2\rho} \frac{(\rho + \rho^{-1})^2}{\rho^{N+1} - \rho^{-(N+1)}} \frac{|d\xi|}{N + 1}.$$

Hence, applying the lemma for  $L_N(z)$  ( $N$  odd), we have the following theorem:

**THEOREM.** *Let  $f \in A[-1, 1]$  and be continuable analytically so as to be regular and single valued in the closed ellipse  $\mathcal{E}_\rho$  with foci at  $z = \pm 1$  and whose sum of the semiaxes is  $\rho$  ( $\rho > 1$ ).*

*Then, from (12),*

$$(26) \quad |E_{N-1}(\psi)| = \frac{1}{\pi} \int_{\mathcal{E}_\rho} \frac{|f(z)| |L_N(z)|}{|T'_{N+1}(z)|} |dz|$$

$$(27) \quad \leq \left[ \frac{\sigma_{N,N+3}}{\rho^2 - 1} + \frac{1}{\rho^{N+1}} \frac{4(N + 1)}{2N + 3} \right] \frac{(\rho + \rho^{-1})^2}{\rho^{N+1} - \rho^{-(N+1)}} M(\rho),$$

where  $M(\rho) = \max_{z \in \mathcal{E}_\rho} |f(z)|$  on  $\mathcal{E}_\rho$  (or equivalently on  $|\xi| = \rho$ ).

*Remarks.* (1) The above estimate is poor for  $\rho$  very nearly equal to 1 and is reasonably good for large  $\rho$ .

(2) It may be seen that for large  $N$ , the contribution from the second term in the bracket of (27) is negligible as compared to the first and hence,

$$(28) \quad |E_{N-1}(\psi)| \leq \frac{\sigma_{N,N+3}}{\rho^2 - 1} \frac{(\rho + \rho^{-1})^2}{\rho^{N+1} - \rho^{-(N+1)}} M(\rho)$$

holds approximately.

(3) To obtain the error estimate of the Clenshaw-Curtis quadrature scheme Chawla [4] uses interpolation points,  $x_i = \cos \pi i/N$ ,  $i = 0(1)N$  ( $N$  even), whereas in the present estimate for the Filippi quadrature method, the corresponding points used are  $x_i = \cos \pi i/N + 1$ ,  $i = 1(1)N$  ( $N$  odd).

Hence, if the same degree of the approximated polynomial is used for the quadrature problem in both cases, the value of  $N$  in the present estimate must exceed by one the corresponding value in Chawla's estimate.

(4) It may be noticed that for large  $N$ , the expression  $16N^2/(4N^2 - 1)$  (which

TABLE 1

$N$	Present estimate	$N$	Chawla's estimate
3	0.0016 6322	2	0.0042 3456
5	0.0000 4028	4	0.0000 8230
7	0.0000 0093	6	0.0000 0166
9	0.0000 00020	8	0.0000 00034

is slightly greater than 4) in Chawla's estimate [4, (24)] corresponds to  $\sigma_{N, N+3}$  in the present estimate (28), the order for the remaining part being the same in both estimates. Also  $\sigma_{N, N+3}$  is of the order  $\log N$  for large  $N$ . Hence, if  $\sigma_{N, N+3} \leq 4$ , the estimate (28) is better than Chawla's. And since the value of  $N$  for which the above inequality holds is quite large, for practical purposes the estimate (28) can be conveniently used to obtain the error estimate in Filippi's quadrature.

**5. Numerical Example.** We now calculate the error estimate (27) for the Chebyshev quadrature for the function  $f(x) = 1/(x + 4)$  in  $[-1, 1]$  and compare the estimates with those obtained by Chawla [4]. We take as in [4, Section 5]  $\rho = 7$ ,  $f(z) = 1/(z + 4)$  on  $\varepsilon_\rho$  and  $M(\rho) = 2.33333347$ . The preceding table represents the estimations of error.

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