Calculation of the Gamma Function by Stirling’s Formula

By Robert Spira

Abstract. In this paper, we derive a simple error estimate for the Stirling formula and also give numerical coefficients.

Stirling’s formula is:

\[ \log \Gamma(s) = (s - \frac{1}{2}) \log s - s + \frac{1}{2} \log 2\pi + \sum_{k=1}^{m} s^{-2k}(2k)^{-1}(2k - 1)^{-1}B_{2k} + R_m \]

where

\[ R_m = - \int_{0}^{\infty} (s + x)^{-2m}(2m)^{-1}B_{2m}(x - [x]) \, dx. \]

Formulas (1) and (2) and a simple estimate for \( |R_m| \) are derived in de Bruijn [1, pp. 46-48].

Another form of \( R_m \), developed on the assumption \( \text{Re} \, s > 0 \), is

\[ R_m = \frac{2(-1)^m}{s^{2m-1}} \int_{0}^{\infty} \left\{ \int_{0}^{t} \frac{u^{2m}}{u^2 + s^2} \right\} \frac{dt}{e^{2s^2} - 1}, \]

(Whittaker and Watson [5, p. 252]), and Whittaker and Watson also estimate this expression, finding

\[ |R_m| \leq \frac{B_{2m+2}}{(2m + 1)(2m + 2)} K(s) \frac{1}{|s|^{2m+1}} \]

where

\[ K(s) = \text{upper bound} \left| s^2/(u^2 + s^2) \right|, \quad u \geq 0. \]

This is the form given in the NBS Handbook, and is clearly poor near the imaginary axis. It follows, however, from this form, that if \( |\arg s| \leq \pi /4 \), then the error in taking the first \( m \) terms of the asymptotic series is less in absolute value than the absolute value of the \((m + 1)\)st term. Another form of the remainder, valid for \( |\arg s| \leq \pi - \delta \), is derived in Whittaker and Watson [5, §13.6], but this remainder involves the Hurwitz zeta function, and has never been used for numerical estimates. An estimate for \( R_m \), as given by (2), may be found in Nielsen [6, p. 208], and, expressed in current notation, is

\[ |R_m(s)| < \frac{B_{2m+2}}{(2m + 1)(2m + 2)} \frac{1}{|s|^{2m+1}} \frac{1}{(\cos (\frac{1}{2} \arg s))^{2m+2}}. \]

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This gives a uniform estimate in the angle $|\arg s| \leq \pi - \delta$. We now develop an estimate for $R_m$ which has the advantages of simplicity in application, and uniformity for a set of points whose distance from the negative real axis is $\geq$ some fixed amount.

**Theorem.**

(3) \[ |R_m| \leq 2 |B_{2m}/(2m - 1)| \cdot |\text{Im } s|^{1-2m} \quad \text{for } \text{Re } s < 0, \text{Im } s \neq 0, \]
(4) \[ |R_m| \leq |B_{2m}/(2m - 1)| \cdot |s|^{1-2m} \quad \text{for } \text{Re } s \geq 0. \]

**Proof.** Since $B_{2m}(x - [x])$ varies only slightly over the range of $x$, and $|B_{2m}(x - [x])| \leq |B_{2m}|$, the problem of estimating $|R_m|$ reduces to the problem of estimating $\int_0^\infty |s + x|^{-2m} \, dx$. Note that the integrand will be large only when $s$ is near $-x$. By symmetry, we need only consider the case when $\text{Im } s \geq 0$. First, let $\text{Re } s < 0$ and $\text{Im } s \neq 0$. Then, taking $k = \text{Im } s$,

\[ \int_0^\infty |s + x|^{-2m} \, dx = \int_0^{-\text{Re } s} + \int_{-\text{Re } s}^{-\text{Re } s + k} + \int_{-\text{Re } s + k}^\infty. \]

Estimating the integrands of the second integral by $|s + x|^{-2m} \leq k^{-2m}$, and of the third by $|s + x|^{-2m} \leq (x + \text{Re } s)^{-2m}$, we obtain

\[ \int_0^\infty |s + x|^{-2m} \, dx \leq \int_0^{-\text{Re } s} |s + x|^{-2m} \, dx + k^{-1-2m} + (2m - 1)^{-1}(k)^{1-2m}. \]

It remains to estimate $\int_{-\text{Re } s}^0$. If $-\text{Re } s \leq k$, we approximate the integrand again by $k^{-2m}$, giving

\[ \int_{-\text{Re } s}^0 |s + x|^{-2m} \, dx \leq (-\text{Re } s) \cdot k^{-1-2m} \leq k^{1-2m}. \]

If, however, $-\text{Re } s > k$, we break up the range of integration, giving

\[ \int_{-\text{Re } s}^0 |s + x|^{-2m} \, dx \leq \int_{-\text{Re } s}^{-\text{Re } s - k} |s + x|^{-2m} \, dx + \int_{-\text{Re } s - k}^{-\text{Re } s} |s + x|^{-2m} \, dx \]

\[ \leq \int_{-\text{Re } s - k}^{-\text{Re } s} (x - \text{Re } s)^{-2m} \, dx + k^{1-2m} \]

\[ = \frac{1}{2m - 1} [k^{1-2m} - (-\text{Re } s)^{-1-2m}] + k^{1-2m} \]

\[ \leq (1 + 1/(2m - 1))k^{1-2m}. \]

So that in all cases, if $\text{Re } s < 0$

\[ \int_0^\infty |x + s|^{-2m} \, dx \leq (4m/(2m - 1))k^{1-2m}, \]

so we have derived (3).

If $\text{Re } s \geq 0$, then $|s + x|^{-2m} \leq |ki + x|^{-2m}$ since

\[ |s + x|^2 = (\text{Re } s + x)^2 + (\text{Im } s)^2 = 2x \text{ Re } s + x^2 + k^2 \geq |ki + x|^2. \]

Next, estimating as before,
\[ \int_0^\infty |k^i + x|^{-2m} \, dx \leq \int_0^k k^{-2m} \, dx + \int_k^\infty x^{-2m} \, dx \leq k^{1-2m}(1 + 1/(2m - 1)), \]

thus giving (4), and completing the proof.

On taking the exponential, we find

\[ \Gamma(s) \sim (2\pi)^{1/2} e^{-s^{1/2}} \exp \left[ \sum_{k=1}^{\infty} \frac{A_{2k-1}}{s^{2k-1}} \right] \]

where

\[ A_{2k-1} = B_{2k}/2k(2k - 1). \]

A short calculation gives (formally)

\[ \exp \left[ \sum_{k=1}^{\infty} \frac{A_{2k-1}}{s^{2k-1}} \right] = 1 + \sum_{k=1}^{\infty} s^{-k} \sum_{\{\alpha_i, \beta_i, \ldots, \alpha_n, \beta_n\} \in Q(k)} \frac{A_{\alpha_1}^{\beta_1} A_{\alpha_2}^{\beta_2} \cdots A_{\alpha_n}^{\beta_n}}{j_1! j_2! \cdots j_n!} \]

\[ = 1 + \sum_{k=1}^{\infty} c_k s^{-k} \]

where the \( \alpha_i \)'s are distinct and \( Q(k) \) is the set of partitions of \( k \) into odd parts (\( \alpha_i^{j_i} \) means \( \alpha_i \) repeated \( j_i \) times in the partition).

Wrench [2] found the recurrences

\[ (2k - 1)c_{2k-1} = \frac{B_2}{2} c_{2k-2} + \frac{B_4}{4} c_{2k-4} + \cdots + \frac{B_{2k}}{2k}, \]

\[ 2kc_{2k} = \frac{B_2}{2} c_{2k-1} + \frac{B_4}{4} c_{2k-3} + \cdots + \frac{B_{2k}}{2k} c_1, \]

where \( k = 1, 2, 3, \cdots \) and \( c_0 = 1 \), and these formulas are more suitable for calculation than (7).

Wrench [2] also gave the \( c_i \)'s for \( j = 0(1)20 \), in exact form and to 50D, and also found approximations to about 6S for \( j = 21(1)30 \). We give in Table 1 the exact rational values for \( j = 21(1)30 \) and in Table 2 their 45D equivalents. The following corrections are necessary in Wrench's tables. In his Table 2, the last ten digits of \( c_{13} \) read 01893 93280, and should read 01894 09396. In his Table 3, entries 22, 23, 24, 26, 28, 30 can be corrected from Table 2 of this paper. Dr. Wrench confirmed the correctness of the author's value for \( c_{13} \), and that it is likely that the author's corrections to his Table 3 are also valid. It is of interest to note that while Dr. Wrench's calculations were carried out on a desk calculator, the author's were performed on a Fortran simulator of a large decimal machine (Spira [7]).

A further calculation revealed that entries 3, 4, 7, 8, 11, 12, 15, 16, 17 for \( c_{n+1}/c_n \) in Table XII of Spira [3] have errors beyond 16S. These errors did not affect the remaining tables.

Finally, we remark that estimates for the error in using

\[ \Gamma(s) \sim (2\pi)^{1/2} e^{-s^{1/2}} \left\{ 1 + \frac{c_1}{s} + \frac{c_2}{s^2} + \cdots + \frac{c_k}{s^k} \right\} \]

can be obtained from estimating

\[ \exp \left\{ \sum_{i=1}^{m} A_{2i-1} s^{1-2i} + R_m \right\} - \sum_{i=1}^{k} c_i s^{-i} \]
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<tr>
<th>$c_{21}$</th>
<th>34856 85173 42344 01648 33562 31076 88675 64083 96794 47003</th>
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<td>2601 64872 18125 16297 62664 73959 14866 28167 68000 00000</td>
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TABLE 2

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<tr>
<th>Coefficients c_{2i}</th>
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<td>29</td>
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<tr>
<td>30</td>
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</table>

and using (3) and (4), where \( m = [(k + 2)/2] \). For example, for \( \Re s \geq 0 \) and \( |s| \geq 1 \), and \( k = m = 2 \), we have

\[
\Gamma(s) = (2\pi)^{1/2}e^{-s^2}e^{-s^2/2} \exp \left\{ \frac{1}{12s} + \frac{1}{360s^3} + R_2 \right\},
\]

where

\[
|R_2| \leq \frac{1}{90|s|^3},
\]

so

\[
|\exp R_2 - 1| \leq |R_2| \{1 + |R_2| + |R_2|^2 + \cdots\} \leq \frac{1}{89|s|^3}.
\]

Next,

\[
\left| \exp \left( \frac{1}{12s} + \frac{1}{360s^3} \right) - \left( 1 + \frac{1}{12s} + \frac{1}{288s^2} \right) \right|
\]

\[
\leq \frac{1}{360|s|^3} + \frac{1}{12 \cdot 360|s|^4} + \frac{1}{2 \cdot 360^2|s|^6} + \frac{1}{3!} \left| \frac{1}{12s} + \frac{1}{360s^3} \right| + \cdots
\]

which estimates as before. Such estimates show the series for \( \Gamma(s) \) is an asymptotic series (de Bruijn [1]).

For calculations near the origin, it is best to use the functional equation \( \Gamma(s + 1) = s\Gamma(s) \) and calculate \( \Gamma(s) = \Gamma(s + j)/P(s) \), where \( P(s) \) is a polynomial. This formula could also be used for larger \( |s| \) for ultraprecise calculations where precisions are needed which are greater than the maximum precision obtainable from the asym-
totic formula. For calculations in the left half-plane with small imaginary part, one can use the equation \( \Gamma(s)\Gamma(1 - s) = \pi/\sin \pi s \).

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