A Fourth-Order Finite-Difference Approximation for the Fixed Membrane eigenproblem*

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Abstract. The fixed membrane problem \( \Delta u + \lambda u = 0 \) in \( \Omega \), \( u = 0 \) on \( \partial \Omega \), for a bounded region \( \Omega \) of the plane, is approximated by a finite-difference scheme whose matrix is monotone. By an extension of previous methods for schemes with matrices of positive type, \( O(h^4) \) convergence is shown for the approximating eigenvalues and eigenfunctions, where \( h \) is the mesh width. An application to an approximation of the forced vibration problem \( \Delta u + qu = f \) in \( \Omega \), \( u = 0 \) in \( \partial \Omega \), is also given.

1. Introduction. Let \( \Omega \) be a bounded region of the plane with smooth boundary \( \partial \Omega \). We consider the fixed membrane problem

\[
\Delta u(x) + \lambda u(x) = 0, \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial \Omega,
\]

where \( \Delta \) is the Laplacian. In [6], this problem was approximated by difference schemes which were of positive type in the interior of the region. Here, we consider a difference scheme for (1.1) which is only monotone. However, by appropriate modifications of the techniques of [6], we can prove that this scheme yields \( O(h^4) \) approximations to the eigenvalues and eigenvectors of (1.1). The principal result is Theorem 8.1. An application to a forced vibration problem is also given in Section 9.

2. The Difference Scheme. Let \( h > 0 \) be given and define the mesh \( S_h \) by

\[
\{(ih, jh) : i, j \text{ are integers}\}.
\]

Points \( x, y \in S_h \) will be called nearest neighbors if \( |x - y| = h \), where we write

\[
|x - y| = ((x_1 - y_1)^2 + (x_2 - y_2)^2)^{1/2}.
\]

Let \( \Omega_h^{(3)} \) be the set of points in \( S_h \cap \Omega \) having at least one nearest neighbor not in \( \Omega \). One such point might be \( x = (x_1, x_2) \) with \( (x_1 - \alpha h, x_2), (x_1, x_2 - \beta h) \in \partial \Omega \) for \( 0 < \alpha, \beta \leq 2 \). If \( (x_1 + h, x_2), (x_1 + 2h, x_2), (x_1, x_2 + h), (x_1, x_2 + 2h) \in \Omega \), we define...
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\[ h^2 l_h(x, y) = \frac{3 - \alpha}{\alpha} + \frac{3 - \beta}{\beta}, \quad y = x, \]

\[ = -\frac{2(2 - \alpha)}{1 + \alpha}, \quad y = (x_1 + h, x_2), \]

\[ = -\frac{2(2 - \beta)}{1 + \beta}, \quad y = (x_1, x_2 + h), \]

\[ = \frac{1 - \alpha}{2 + \alpha}, \quad y = (x_1 + 2h, x_2), \]

\[ = \frac{1 - \beta}{2 + \beta}, \quad y = (x_1, x_2 + 2h), \]

\[ = 0, \quad \text{otherwise}. \]

(2.1)

Similar formulas apply at other points of \( \Omega_h^{(3)} \). One special case may arise, as shown in Fig. 1, where \((x_1, x_2 + h), (x_1, x_2 + 2h)\) do not lie in \( \Omega \).

![Figure 1](image_url)

In such a case \( x \) would be excluded from the difference scheme altogether and the point \((x_1 + h, x_2)\) would be added to \( \Omega_h^{(3)} \). For the new point, formula (2.1) would be used with \( 1 < \alpha \leq 2 \). If \( \partial \Omega \) has bounded curvature and \( h \) is sufficiently small, there will be no difficulty with the new point.

Next, let \( \Omega_h^{(2)} \) be those points of \( S_h \cap \Omega \), not in \( \Omega_h^{(3)} \) or excluded, which have a nearest neighbor in \( \Omega_h^{(3)} \). For \( x \in \Omega_h^{(2)} \) define

\[ h^2 l_h(x, y) = \begin{cases} 4, & y = x, \\ -1, & |x - y| = h, \ y \in S_h, \\ 0, & \text{otherwise}. \end{cases} \]

(2.2)

Finally, let \( \Omega_h' \) be those points of \( S_h \cap \Omega \) not in \( \Omega_h^{(2)} \cup \Omega_h^{(3)} \) or excluded. For \( x \in \Omega_h' \) define

\[ h^2 l_h(x, y) = \begin{cases} 5, & y = x, \\ -\frac{3}{4}, & |x - y| = h, \ y \in S_h, \\ \gamma_{h}, & |x - y| = 2h, \ y \in S_h, \\ 0, & \text{otherwise}. \end{cases} \]

(2.3)
Let \( \Omega_h = \Omega_h^1 \cup \Omega_h^{(2)} \cup \Omega_h^{(3)} \). We approximate the Laplacian of a function \( u \) vanishing on \( \partial \Omega \) by
\[
-\Delta_h u(x) = \sum_{y \in \partial \Omega_h} I_h(x, y) u(y), \quad x \in \Omega_h.
\]

Let us agree to use \( C \) as a generic constant, whose value may change at each usage, but which is always independent of \( h \). Then, if also \( u \in C^6(\Omega) \) (\( u \) has continuous sixth derivatives on the closure of \( \Omega \)), it can be seen from Taylor series expansions that
\[
|\Delta u(x) - \Delta_h u(x)| \leq Ch^4, \quad x \in \Omega_h,
\]
\[
\leq Ch^2, \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}.
\]

Bramble and Hubbard used \( \Delta_h \) in [2] in approximating the Dirichlet problem for Poisson’s equation.

Our difference scheme approximating (1.1) is
\[
\Delta_h U_h(x) + \lambda_h U_h(x) = 0, \quad x \in \Omega_h.
\]

Problem (2.6) is equivalent to finding the eigenvalues and eigenvectors of the matrix \([I_h(x, y)]_{x,y \in \Omega_h}\). In the next section, we develop some tools to use in studying this matrix which, however, have some independent interest.

### 3. Monotone Matrices

Let \( A = (a_{ij}) \) be an \( n \times n \) matrix. We say \( A \succeq 0 \) if each \( a_{ii} \geq 0 \) and \( A \preceq B \) if \( B - A \succeq 0 \). The matrix \( A \) is monotone if \( Ax \succeq 0 \) implies \( x \succeq 0 \) for all \( x \). Thus, \( A \) is monotone if and only if \( A^{-1} \) exists and \( A^{-1} \succeq 0 \). An easily recognized type of monotone matrix is a matrix of positive type. The matrix \( A \) is of positive type if \( A \) is indecomposable, the diagonal of \( A \) is positive, the off-diagonal elements negative, and the row sums are nonnegative with at least one strictly positive. The following theorem is due to Price [8]:

**Theorem 3.1.** \( A \) is monotone if and only if there exists \( M \) monotone such that
\begin{enumerate}[(i)]  
  \item \( M^{-1}(M - A) \succeq 0 \),
  \item \( \rho(M^{-1}(M - A)) < 1 \).
\end{enumerate}

Here \( \rho \) denotes spectral radius, the maximum of the moduli of the eigenvalues. Here and in the corollaries, the “only if” part is trivial: take \( M = A \). This theorem generalizes Theorem 2.7 of Bramble and Hubbard [2]. There are a number of important corollaries:

**Corollary 3.2.** \( A \) is monotone if and only if there exists \( M \) monotone such that
\begin{enumerate}[(i)]  
  \item \( M \succeq A \),
  \item \( \rho(M^{-1}(M - A)) < 1 \).
\end{enumerate}

**Corollary 3.3.** \( A \) is monotone if and only if there exists \( M \) monotone and \( x > 0 \) such that
\begin{enumerate}[(i)]  
  \item \( M \succeq A \),
  \item \( Ax > 0 \).
\end{enumerate}

**Proof.** By the Gerschgorin circle theorem (see [7, p. 152]),
\[
\rho(M^{-1}(M - A)) \leq \max_i [M^{-1}(M - A)x_i]/x_i < 1,
\]
since
\[
0 \leq [M^{-1}(M - A)x_i]i = x_i - [M^{-1}Ax_i]i < x_i,
\]
because \( Ax > 0 \), \( M^{-1} \succeq 0 \) and no row of \( M^{-1} \) can be all zero.
Corollary 3.4. A is monotone if and only if there exists $M$ monotone and $x \geq 0$ such that

(i) $M \geq A$,
(ii) $Ax > 0$.

Proof. Let $\delta = \min_i |Ax|, > 0$ and let $\epsilon = \delta/(2 \max_i \sum_i |a_{i,j}|).$ Then $x + \epsilon > 0$ and $A(x + \epsilon) > 0$, so the hypotheses of Corollary 3.4 are satisfied.

Corollary 3.5. A is monotone if and only if there exist $M_1, M_2$ monotone such that

$M_1 \leq A \leq M_2$.

Proof. Let $x$ be such that $M_1 x$ is the vector with all components 1. Since $M_1$ is monotone, $x$ exists and $x \geq 0$. Also, $Ax \geq M_1 x > 0$, so the hypotheses of Corollary 3.4 are satisfied.

Corollary 3.6. A is monotone if there is $\alpha > 0$ such that $A + \alpha I$ is monotone and every eigenvalue $\lambda$ of $A$ has positive real part.

Proof. Apply Corollary 3.2. We need only show $\rho((A + \alpha I)^{-1}) < \alpha^{-1}$. But $\rho((A + \alpha I)^{-1}) = 1/\min_{\lambda} |\alpha + \lambda|$, where $\lambda$ runs over the eigenvalues of $A$.

At this time, we also note the following:

Lemma 3.7. If the partitioned matrix

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]

with $A$ invertible has inverse

\[
\begin{bmatrix}
W & X \\
Y & Z
\end{bmatrix}
\]

then $W - A^{-1} = -XCA^{-1}$. In particular, if $X \geq 0, A^{-1} \geq 0, C \leq 0$, then $A^{-1} \leq W$.

Proof. Since

\[
\begin{bmatrix}
W & X \\
Y & Z
\end{bmatrix} \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix},
\]

we have $WA + XC = I$. Multiply on the right by $A^{-1}$.

4. Discrete Green’s Functions. The main tools in our investigations will be discrete analogues of Green’s function. These are inverses of matrices related to $[\mathbf{h}_2 l_h(x, y)]_{x, y \in a_k}$ and their nonnegativity is crucial to the investigation. This will be established, using results of the previous section.

We define then

\[\Delta h, s g_h(x, y) = h^{-2} \delta(x, y), x \in \Omega_k \cup \Omega^{(2)}_h, \quad g_h(x, y) = \delta(x, y), x \in \Omega^{(3)}_h,\]

for all $y \in \Omega_k$. This is the discrete Green’s function considered by Bramble and Hubbard in [2, Eq. (4.5)]. From (4.1), we see that the matrix $[g_h(x, y)]_{x, y \in a_k}$ is the inverse of the partitioned matrix

\[
\mathbf{M} = \begin{bmatrix}
A & B \\
0 & I
\end{bmatrix},
\]
where $A = [h^2 I(x, y)]_{x, y \in \Omega_{k^3} \cup \Omega_{k^2}}$, $B = [h^2 I(x, y)]_{x \in \Omega_{k^3} \cup \Omega_{k^2}, y \in \Omega_{k^2}}$, and $I$ is the identity on $\Omega_{k^3} \times \Omega_{k^2}$. It also follows from Lemma 3.7 that the matrix 
$[g_a(x, y)]_{x, y \in \Omega_{k^3} \cup \Omega_{k^2}}$ is the inverse of $A$. In [2], it was shown that

(4.2) $g_a(x, y) \geq 0, \quad x, y \in \Omega_k,$

i.e., $\Omega$ is monotone. Since $g_a$ is the inverse, it follows that, for any function $W$ defined on $\Omega_k$, all $x \in \Omega_k$,

(4.3) $W(x) = h^2 \sum_{y \in \Omega_{k^3} \cup \Omega_{k^2}} g_a(x, y)[- \Delta_k W(y)] + \sum_{y \in \Omega_{k^2}} g_a(x, y) W(y).$

This is analogous to Poisson's formula. In [2], the following properties were proved of $g_a$:

(4.4) $\sum_{y \in \Omega_{k^3} \cup \Omega_{k^2}} g_a(x, y) \leq 1,$

(4.5) $\sum_{y \in \Omega_{k^2}} g_a(x, y) \leq C,$

(4.6) $h^2 \sum_{y \in \Omega_{k^3}} g_a(x, y) \leq C,$

for all $x \in \Omega_k$. Using these in (4.3), we have the inequality

(4.7) $\max_{\Omega_k} |W| \leq C \left[ \max_{\Omega_k} |\Delta_k W| + h^2 \max_{\Omega_k} |\Delta_k W| \right] + \max_{\Omega_k} |W|.$

Now, on $\Omega_{k^3}$, we have

$W(x) = \left[-h^2 \Delta_k W(x) - h^2 \sum_{y \in \Omega_{k^3} \cup \Omega_{k^2}} l_a(x, y) W(y) \right] / h^2 l_a(x, x),$

and from this and (2.1), we see that

(4.8) $\max_{\Omega_{k^2}} |W| \leq C h^2 \max_{\Omega_{k^2}} |\Delta_k W| + \theta \max_{\Omega_k} |W|,$

where

$\theta = \max_{x \in \Omega_{k^3} \cup \Omega_{k^2}} \sum_{y \in \Omega_{k^3} \cup \Omega_{k^2}} \left|l_a(x, y) / l_a(x, x)\right| < 1.$

Putting (4.8) into (4.7) and rearranging, we have

(4.9) $\max_{\Omega_k} |W| \leq C \left[ \max_{\Omega_k} |\Delta_k W| + h^2 \max_{\Omega_{k^2} \cup \Omega_{k^2}} |\Delta_k W| \right].$

Let us now use (4.7) to estimate $W = \Phi_k - \varphi$ where $\varphi$ is the torsion function defined by $\Delta \varphi = -1$ on $\Omega$, $\varphi = 0$ on $\partial \Omega$ and $\Phi_k(x) = h^2 \sum_{y \in \Omega_{k^2}} g_a(x, y)$, which satisfies $\Delta_k \Phi_k = -1$ on $\Omega_{k^3} \cup \Omega_{k^2}$. If $\partial \Omega$ is sufficiently smooth, $\varphi$ satisfies (2.5) and we see from (4.7) that

$\max_{\Omega_k} |\Phi_k - \varphi| \leq C h^4 + \max_{\Omega_{k^2} \cup \Omega_{k^2}} |\Phi_k - \varphi| \leq C h^4 + \max_{\Omega_{k^2}} |\Phi_k| + \max_{\Omega_{k^2}} |\varphi|.$

Now, $\varphi = 0$ on $\partial \Omega$, so $|\varphi(x)| \leq C h$ for $|x - \partial \Omega| = \min_{x \in \partial \Omega} |x - y| \leq C h$. Also, $\Phi_k = h^2$ on $\Omega_{k^2}$ by definition. Hence,

$|\Phi_k(x)| \leq |\varphi(x)| + \max_{\Omega_k} |\Phi_k - \varphi| \leq C h.$
for $|x - \partial \Omega| \leq Ch$, i.e.,

$$h^2 \sum_{y \in \Omega_h} g_h(x, y) \leq Ch.$$  

Next, we consider the function

$$f_h(x, y) = C_1 - C_2 \log (|x - y|^2 + h^2).$$

It is easily verified that

$$-\Delta_h f_h(x, y) \geq 0, \quad x \in \Omega_h' \cup \Omega_h^{(2)}, \quad y \neq x,$$

$$-\Delta_h f_h(x, y) \geq h^{-2}, \quad x \in \Omega_h' \cup \Omega_h^{(2)}, \quad y = x,$$

provided $C_2 \geq \tfrac{1}{2} \log 2$. If we choose

$$C_1 = C_2 \max_{x, y \in \Omega_h} \log (|x - y|^2 + h^2),$$

then $f_h(x, y) \geq 0$ for $x, y \in \Omega$. Thus, we see that

$$\mathcal{M}(f_h - g_h) \geq 0,$$

and, since $\mathcal{M}$ is monotone,

$$0 \leq g_h(x, y) \leq C_1 - C_2 \log (|x - y|^2 + h^2).$$

Analogous inequalities to (4.11) are proved by Bramble and Thomée in [3] for discrete Green's functions of positive-type operators. Here, we see monotonicity was sufficient.

An easy consequence of (4.11) is

$$h^2 \sum_{y \in \Omega_h} [g_h(x, y)]^2 \leq C.$$

5. More Inequalities for Green's Functions. This section will be devoted to derivations of some inequalities of more difficulty than those of the previous section. Recall that $\mathcal{O} = [g_h(x, y)]_{x, y \in \Omega_h}$ is the inverse of $[h^2 f_h(x, y)]_{x, y \in \Omega_h}$.

The inequality which we next wish to derive is

$$(5.1) \quad \sum_{x \in \Omega_h} g_h(x, y) \leq C$$

for all $x \in \Omega_h$, where $\Omega_h' = \{x \in \Omega_h'; f_h(x, y) \neq 0 \text{ for some } y \in \Omega_h^{(2)} \cup \Omega_h^{(3)}\}$. The method of proof is the matrix splitting technique employed by Bramble and Hubbard in [2]. The analysis which follows is regrettably detailed.

Let us write

$$(5.2) \quad \mathcal{O} = [I - H_1 - H_2]^{-1} \tilde{D}^{-1},$$

where $\tilde{D}$ is the diagonal matrix with

$$\tilde{d}_{xx}^{-1} = 1, \quad x \in \Omega_h^{(3)},$$

$$= \tfrac{1}{3}, \quad x \in \Omega_h^{(2)},$$

$$= \tfrac{1}{5}, \quad x \in \Omega_h.$$
and

\[ [H_1]_{xy} = \begin{cases} 0, & x \in \Omega_h', x \not\in \Omega_h, |x - y| = h, \\ \frac{1}{h}, & x \in \Omega_h^{(2)}, x \not\in \Omega_h, |x - y| = h, \\ 0, & \text{otherwise}, \end{cases} \]

\[ [H_2]_{xy} = \begin{cases} 0, & x \in \Omega_h', x \not\in \Omega_h, |x - y| = h, \\ -\frac{1}{h}, & x \in \Omega_h^{(2)}, x \not\in \Omega_h, |x - y| = 2h, \\ \frac{1}{h}, & \text{otherwise}. \end{cases} \]

Let us define the diagonal matrix \( D \) by

\[
(d_{xx})^{-1} = \sum_{y \in \Omega_h} (I - H)_y = \frac{1}{h}, \quad x \in \Omega_h',
\]

\[
= \frac{1}{h}, \quad x \in \Omega_h^{(2)},
\]

\[
= 1, \quad x \in \Omega_h^{(3)},
\]

so that \( D(I - H) \) has row sums one, i.e.,

\[
\sum_{y \in \Omega_h} [D(I - H)]_{xy} = \sum_{y \in \Omega_h} [(I - H)^{-1}D^{-1}]_{xy} = 1. \tag{5.3}
\]

We write \([I - H_1 - H_2] = [D^{-1}(I - H)]D(I - H_1)]\), where \( H = DH_2(I - H_1)^{-1}D^{-1} \).

Thus, by (5.3),

\[
\sum_{y \in \Omega_h} [D^{-1}(I - H)]_{xy} = \sum_{y \in \Omega_h} [D^{-1}(I - H)]_{xy}D(I - H_1)]_{xy}
\]

\[
\sum_{x \in \Omega_h} [I - H_1 - H_2]_{xx} = 0, \quad x \in \Omega_h' \cup \Omega_h^{(2)},
\]

\[
= 1, \quad x \in \Omega_h^{(3)}. \tag{5.4}
\]

Now, we consider the characteristic function of \( \Omega_h' \):

\[
\chi(x) = \begin{cases} 1, & x \in \Omega_h', \\ 0, & x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}. \end{cases}
\]

Then

\[
1 \geq \chi(x) = \left[ ((I - H)^{-1}D)[D^{-1}(I - H)\chi]\right]_x
\]

\[
= \sum_{y \in \Omega_h} [(I - H)^{-1}D]_{xy}[D^{-1}(I - H)\chi]_x
\]

\[
+ \sum_{y \in \Omega_h^{(2)} \cup \Omega_h^{(3)}} [(I - H)^{-1}D]_{xy}[D^{-1}(I - H)\chi]_x
\]

\[
= \sum_{y \in \Omega_h} [(I - H)^{-1}D]_{xy} \sum_{x \in \Omega_h} [D^{-1}(I - H)]_{xx}
\]

\[
- \sum_{y \in \Omega_h} [(I - H)^{-1}D]_{xy} [D^{-1}(I - H)(1 - \chi)]_x
\]

\[
+ \sum_{y \in \Omega_h^{(2)} \cup \Omega_h^{(3)}} [(I - H)^{-1}D]_{xy} [D^{-1}(I - H)\chi]_x.
\]
By (5.4), the first term vanishes. Using the definitions of $H$ and $\chi$, this can be written as

$$
\sum_{y \in \Omega} [(I - H)^{-1}D]_{xy} \sum_{y \in \Omega} (H_3(I - H_1)^{-1}D^{-1})_{yx}
$$

(5.5)

$$
- \sum_{y \in \Omega \cap \Omega_4} [(I - H)^{-1}D]_{xy} \sum_{y \in \Omega} (H_3(I - H_1)^{-1}D^{-1})_{yx} \leq 1.
$$

Now, we estimate the factors in each term. First, note that $(I - H)^{-1} \geq 0$. This is not obvious, but follows from $H \geq 0$ and $\rho(H) < 1$. That $H \geq 0$, is due to $0 \leq H_3(I - H_1)^{-1} = H_2 + H_2H_1 + \cdots$, since the negative terms in $H_2$ are cancelled by positive terms in $H_2H_1$ as in [2]. That $\rho(H) < 1$ is due to $\rho(H) = \rho((I - H_1)^{-1}H_2) < 1$, since the row sums of

$$(I - (I - H_1)^{-1}H_2) = (I - H_1)^{-1}(I - H_1 - H_2)$$

$$= (I - H_1 - H_2) + H_1(I - H_1 - H_2) + \cdots$$

are positive. Again negative row sums of $(I - H_1 - H_2)$ are cancelled by corresponding positive row sums of $H_1(I - H_1 - H_2)$.

Next, for $y \in \Omega_2 \cup \Omega_4$

$$
\sum_{y \in \Omega} (H_2(I - H_1)^{-1}D^{-1})_{yx} \leq \sum_{y \in \Omega} (D^{-1} - D^{-1}(I - H))_{yx} \leq 1 - \sum_{y \in \Omega} (I - H_1 - H_2)_{yx} \leq 1.
$$

Now, we consider, for $y \in \Omega_3'$, the term

$$
(5.6) \sum_{y \in \Omega} (H_2(I - H_1)^{-1}D^{-1})_{yx}.
$$

Expanding the summand in a Neumann series, it becomes

$$
[(H_2 + H_2H_1 + H_2H_1^2 + \cdots)D^{-1}]_{yx}.
$$

If $y \in \Omega_3'$, $z \in \Omega_2 \cup \Omega_4$ is such that $|y - z| = 2h$, then $[H_2]_{yx} = -1/60$. However, let $x$ be the point such that $|y - x| = |x - z| = h$. Then $[H_2H_1]_{yx}$ contains the term $[H_2]_{yx}[H_1]_{yx} = 4/225$. Similarly, each negative term in $H_2H_1^2$ is compensated for by a positive term in $H_2H_1^{2t}$, Thus, for $y \in \Omega_3'$,

$$
\sum_{y \in \Omega} (H_2(I - H_1)^{-1}D^{-1})_{yx} \geq \left(-\frac{1}{60} + \frac{4}{225}\right) \frac{1}{2} = \frac{1}{1800}.
$$

It follows from (5.5) and the above that

$$
(5.7) \sum_{y \in \Omega} [(I - H)^{-1}D]_{xy} \leq 1800 \left(1 + \sum_{y \in \Omega} (H_2(I - H_1)^{-1}D^{-1})_{yx}\right).
$$

By similar reasoning, using the function

$$
\chi(x) = 1, \quad x \in \Omega_4 \cup \Omega_2 \cup \Omega_4 \cup \Omega_4,
$$

$$
= 0, \quad x \in \Omega_2 \cup \Omega_4 \cup \Omega_3 \cup \Omega_4,
$$

it can be shown that $\sum_{y \in \Omega} [(I - H)^{-1}D]_{xy} \leq C$. The argument is carried out in [2, Lemma 3.3]. Finally, we note from (5.4) that
Combining the above with (5.7), we see that
\[
(5.9) \quad \sum_{\nu \in \Omega_k} [(I - H)^{-1} D]_{\nu y} \leq C.
\]
From (5.2) and (5.3), we finally have
\[
(5.10) \quad \sum_{\nu \in \Omega_k} g_k(x, y) = \sum_{\nu \in \Omega_k} [(I - H_1 - H_2)^{-1} D^{-1}]_{\nu y} = \frac{1}{3} \sum_{\nu \in \Omega_k} [(I - H_1 - H_2)^{-1}]_{\nu y}
\]
\[
= \frac{1}{3} \sum_{\nu \in \Omega_k} \sum_{\nu \in \Omega_k} [(D(I - H_1)]^{-1}]_{\nu y} [(I - H)^{-1} D]_{\nu y}
\]
\[
\leq \frac{1}{3} \max_{\nu \in \Omega_k} \sum_{\nu \in \Omega_k} [(I - H)^{-1} D]_{\nu y},
\]
or, from (5.9),
\[
(5.10) \quad \sum_{\nu \in \Omega_k} g_k(x, y) \leq C,
\]
the desired estimate.

We next define another Green's function \( G_k \) by
\[
(5.11) \quad -\Delta_k G_k(x, y) = \lambda_k^2 \delta(x, y), \quad x, y \in \Omega_k.
\]
Although \( G_k \) may not be nonnegative, it is a perturbation of \( g_k \). We have

**Theorem 5.1.** For any mesh function \( S \),
\[
(5.12) \quad \max_{x \in \Omega_k} \sum_{\nu \in \Omega_k} |[G_k(x, y) - g_k(x, y)] S(y)| \leq C \max_{\nu \in \Omega_k} |S| + \max_{\nu \in \Omega_k} \sum_{\nu \in \Omega_k} \sum_{\nu \in \Omega_k} g_k(x, y) |S(y)|.
\]

**Proof.** Let \( x_0 \in \Omega \) be the point where \( \sum_{\nu \in \Omega_k} |[G_k(x, y) - g_k(x, y)] S(y)| \) attains its maximum and let
\[
W(x) = \sum_{\nu \in \Omega_k} [G_k(x, y) - g_k(x, y)] S^*(y),
\]
where \( S^*(y) = |S(y)| \text{ sgn} [G_k(x_0, y) - g_k(x_0, y)] \), Employing (4.9), we have
\[
\max_{\Omega_k} |W| \leq C \max_{\nu \in \Omega_k} |h^2 \Delta_k W|
\]
\[
\leq C \left[ \max_{\nu \in \Omega_k} |S| + \max_{\nu \in \Omega_k} \sum_{\nu \in \Omega_k} \sum_{\nu \in \Omega_k} g_k(x, y) S^*(y) \right],
\]
and (5.12) follows.

**Corollary 5.2.** For all \( x, z \in \Omega_k \),
\[
(5.13) \quad \sum_{\nu \in \Omega_k} |G_k(x, y)| \leq C,
\]
\[
(5.14) \quad h^2 \sum_{\nu \in \Omega_k} |G_k(x, y)| \leq C,
\]
\[
(5.15) \quad |G_k(x, z)| \leq C \log h,
\]
\[
(5.16) \quad h^2 \sum_{\nu \in \Omega_k} |G_k(x, y)|^2 \leq C,
\]

and for $|x - \partial \Omega| \leq C h$,

\begin{equation}
(5.17) \quad h^2 \sum_{y \in \Omega_h} |G_h(x, y)| \leq C h.
\end{equation}

**Proof.** For (5.13), employ the characteristic function of $\Omega_h^* \cup \Omega_h^{(2)} \cup \Omega_h^{(3)}$ as $S$ in (5.13). Then apply the triangle inequality and (4.4), (4.5), and (5.10). For (5.14), let $S = h^2$ and use (4.6) and (4.10), respectively. For (5.15), let $S(y) = \delta(y, z)$ in (5.12), apply the triangle inequality and (4.11). For (5.16), let $x_0$ be the point where $\max_{x \in \Omega_h} h^2 \sum_{y \in \Omega_h} |G_h(x, y)|^2$ is attained, and let $S(y) = h^2 G_h(x_0, y)$ in (5.12), from which it follows that

\[ h^2 \sum_{y \in \Omega_h} |G_h(x, y)|^2 \leq C h^2 \max_{x \in \Omega_h} |G_h(x, y)| + \max_{x \in \Omega_h} h^2 \sum_{y \in \Omega_h} g_h(x, y) G_h(x_0, y). \]

Again, using (5.12) with $S(y) = h^2 g_h(x, y)$ for $x$ fixed,

\[ h^2 \sum_{y \in \Omega_h} G_h(x_0, y) g_h(x, y) \leq C h^2 \max_{x \in \Omega_h} |g_h(x, y)| + \max_{x \in \Omega_h} h^2 \sum_{y \in \Omega_h} g_h(x, y) g_h(x, y). \]

By (4.11), this term can be seen to be bounded. Finally, letting $S(y) = h^2 \delta(y_0, y)$ in (5.12), we have, for any $y_0 \in \Omega_h$,

\[ |h^2 G_h(x_0, y_0)| \leq C \left[ h^2 + \max_{x \in \Omega_h} h^2 g_h(x, y_0) \right], \]

which indeed tends to zero as $h$ does, by (4.11), and (5.16) follows. For (5.17) use $S = h^2$ and (4.10).

We require yet one more Green’s function $G'_h$ defined by

\begin{equation}
(5.18) \quad -\Delta_h G'_h(x, y) = h^2 \delta(x, y), \quad x \in \Omega_h^*, \quad G'_h(x, y) = 0, \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)},
\end{equation}

for all $y \in \Omega_h$. Thus, the matrix $[G'_h(x, y)]_{x \in \Omega_h^*, y \in \Omega_h^*}$ is the inverse of the symmetric matrix $\mathcal{A} = [h^2 h(x, y)]_{x \in \Omega_h^*, y \in \Omega_h^*}$. We show $\mathcal{A}$ is monotone by applying Corollary 3.6. First, we show $\mathcal{A} + \frac{1}{2} I$ monotone from Corollary 3.5: we define $M_1$ by

\[ [M_1]_{x y} = \begin{cases} \frac{16}{3}, & x = y, \\ -\frac{4}{3}, & |x - y| = h, \\ 0, & \text{otherwise}, \end{cases} \]

for $x, y \in \Omega_h^*$, and we define

\[ [M_2]_{x y} = \begin{cases} \frac{8}{\sqrt{12}}, & x = y, \\ -\frac{1}{\sqrt{12}}, & |x - y| = h, \\ 0, & \text{otherwise}. \end{cases} \]

Since $M_1$ and $M_2$ are of positive type, they are monotone, hence, so is $M_2^*$, and it is easy to see that

\[ M_1 \leq \mathcal{A} + \frac{1}{2} I \leq M_2^*. \]
Thus, \( \mathcal{E} \) is monotone if its eigenvalues, necessarily real by symmetry, are positive. But these are \( h^2 \mu_h^{(i)} \), where \( \mu_h^{(i)} \) is the \( i \)th eigenvalue satisfying

\[
\Delta_h V_h^{(i)}(x) + \mu_h^{(i)} V_h^{(i)}(x) = 0, \quad x \in \Omega_h, \quad V_h^{(i)}(x) = 0, \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}.
\]

In the next section, we shall show that indeed \( |\mu_h^{(i)} - \lambda^{(i)}| \to 0 \) as \( h \to 0 \), for \( \lambda^{(i)} \) the \( i \)th eigenvalue of (1.1), which is strictly positive. Thus, for \( h \) sufficiently small, \( \mathcal{E} \) is monotone and \( G_t' \) nonnegative. Thus, as a consequence of Lemma 3.7,

\[
(5.20) \quad 0 \leq G_t'(x, y) \leq g_h(x, y), \quad x, y \in \Omega_h.
\]

From (5.20), we see that all of the inequalities proved for \( g_h \) hold for \( G_t' \). In particular, the difficult inequality (5.10) does, from which we prove the key inequality

\[
(5.21) \quad \max_{\Omega_h} |W| \leq C \left[ \max_{\Omega_h} |\Delta_h W| + \max_{\Omega_h^{(2)} \cup \Omega_h^{(3)}} |W| \right],
\]

for all \( W \) defined on \( \Omega_h \). To prove this, let

\[
W^*(x) = W(x), \quad x \in \Omega_h',
\]

\[
= 0, \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}.
\]

Then, by (5.18),

\[
W^*(x) = h^2 \sum_{y \in \Omega_h'} G_t'(x, y)[-\Delta_h W^*(y)]
\]

\[
= h^2 \sum_{y \in \Omega_h'} G_t'(x, y)[-\Delta_h W(y)] + h^2 \sum_{y \in \Omega_h'} G_t'(x, y)[\Delta_h W(y) - \Delta_h W^*(y)],
\]

and (5.21) follows from (4.6), (5.10), and (5.20).

6. Convergence of \( \mu_h^{(n)} \) to \( \lambda^{(n)} \). In this section, we show that the eigenvalue \( \mu_h^{(n)} \) of

\[
(6.1) \quad \Delta_h V_h^{(n)}(x) + \mu_h^{(n)} V_h^{(n)}(x) = 0, \quad x \in \Omega_h', \quad V_h^{(n)}(x) = 0, \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)},
\]

converges to \( \lambda^{(n)} \) of (1.1) for each \( n \). We will use the variational principles associated with (1.1) and (6.1), and a technique of Weinberger [9].

The \( n \)th eigenvalue of (1.1) can be characterized by

\[
(6.2) \quad \lambda^{(n)} = \min \max D(u) \int_\Omega u^2 \, dx,
\]

where \( u = \alpha_1 U_1 + \cdots + \alpha_n U_n \), the max is with respect to the scalars \( \alpha_1, \ldots, \alpha_n \), the min is with respect to choices of linearly independent \( u_1, \ldots, u_n \), continuous, piecewise differentiable functions vanishing on \( \partial \Omega \), and \( D(u) \) is the Dirichlet integral.

Similarly, the \( n \)th eigenvalue of (6.1) can be characterized by

\[
(6.3) \quad \mu_h^{(n)} = \min \max \frac{h^2 \sum \left[ U_{x_i}^2 + U_{x_i}^2 + \frac{h^2}{12} U_{x_i x_i}^2 + \frac{h^2}{12} U_{x_i x_i}^2 \right]}{h^2 \sum \ell^2},
\]

where \( \ell = \alpha_1 U_1 + \cdots + \alpha_n U_n \), the max is with respect to the scalars \( \alpha_1, \ldots, \alpha_n \), the min is with respect to choices of linearly independent mesh functions \( U_1, \ldots, U_n \) vanishing on \( \Omega_h^{(2)} \cup \Omega_h^{(3)} \), the sum is over the mesh points of \( \Omega_h \), and subscript \( x_i \).
denotes forward (backward) difference quotient in the \(x_i\) direction, \(i = 1, 2,\) i.e.,

\(U_{x_i}(y_1, y_2) = [U(y_1 + h, y_2) - U(y_1, y_2)]/h,\) etc.

First, we show

\[(6.4) \quad \mu_h^{(n)} \leq \lambda^{(n)} + O(h).\]

Let \(u^{(1)}, \ldots, u^{(n)}\) be eigenfunctions associated with \(\lambda^{(1)}, \ldots, \lambda^{(n)}\) in (1.1), \(u = \alpha_1 u^{(1)} + \cdots + \alpha_n u^{(n)}\), and define

\[u(x) = h^{-1} \int_{Q_h(x)} u(y) \, dy, \quad x \in \Omega_h',\]

\[= 0, \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)},\]

where \(Q_h(x) = \{(y_1, y_2): |x_1 - y_1| \leq \frac{1}{2}h, |x_2 - y_2| \leq \frac{1}{2}h\}\) is the square of side \(h\) centered at \(x\). Put this \(U\) in (6.3). Employing inequalities (2.14), (2.22) and (8.6) of Weinberger [9], we see that

\[p_k n \leq \max_{\alpha} \frac{h^2}{12} \int_0 \left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2 \, dx,\]

and Hubbard [5, pp. 568–569], has shown

\[\int_0 \left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2 \, dx \leq C(\lambda^{(n)})^2.\]

From these, (6.4) follows.

Next, we show

\[(6.5) \quad \lambda^{(n)} \leq \mu_h^{(n)} + O(h).\]

Let \(V^{(1)}_h, \ldots, V^{(n)}_h\) be eigenvectors associated with \(\mu_h^{(1)}, \ldots, \mu_h^{(n)}\) in (6.1), \(U = \alpha_1 V^{(1)}_h + \cdots + \alpha_n V^{(n)}_h\), and define \(u\) to be the continuous, piecewise linear function interpolating \(U\) (see [9, Section 6]). Then, by (6.4), (6.7) of [9] we see that

\[\lambda^{(n)} \leq \max_{\alpha} \frac{h^2}{12} \sum \frac{U_{x_i}^2 + U_{x_i x_i}^2}{h^2} \sum \left(U_{x_i}^2 + U_{x_i x_i}^2\right)\]

\[\leq \max_{\alpha} \frac{h^2}{12} \sum \left[U_{x_i}^2 + U_{x_i x_i}^2 + \frac{h^2}{12} U_{x_i x_i}^2, + \frac{h^2}{12} U_{x_i x_i}^2\right]

\[= \frac{\mu_h^{(n)}}{1 - \frac{h^2}{12} \mu_h^{(n)}}\]

and we obtain (6.5). Combining (6.4) and (6.5), we have

\[(6.6) \quad |\mu_h^{(n)} - \lambda^{(n)}| \to 0 \quad \text{as} \quad h \to 0,\]

for each \(n = 1, 2, \ldots\).
7. Convergence of \( \lambda_i^{(n)} \) to \( \lambda_i^{(n)} \) by Perturbation. Next, we will show that the \( \lambda_i^{(n)} \) are a perturbation of the \( \lambda_i^{(n)} \), and that as \( h \) tends to zero, \( \lambda_i^{(n)} \) tends to \( \mu_i^{(n)} \), hence to \( \lambda_i^{(n)} \), by Section 6. We employ the following theorem of Wielandt:

**Theorem 7.1.** If \( A, B \) are \( \nu \times \nu \) matrices and \( A \) has an orthonormal basis of eigenvectors, then the eigenvalues of \( B \) lie in the union of the \( \nu \) discs \( |\mu - z| \leq ||A - B||_2 \), where the \( \mu_i \) are the eigenvalues of \( A \). If \( k \) discs are disjoint from the others, they contain exactly \( k \) eigenvalues of \( B \).

In the theorem, \( ||\cdot||_2 \) is the spectral norm of a matrix, defined by

\[
||M||_2 = \sup \{ ||M\xi||_2 / ||\xi||_2 \}, \quad \text{where} \quad ||\xi||_2 = \left( \sum_{i=1}^\nu |\xi_i|^2 \right)^{1/2}
\]

for a \( \nu \)-vector \( \xi = (\xi_1, \cdots, \xi_\nu) \). For a proof of the theorem, see [6].

We apply the theorem as follows. For \( A \), we take the matrix \([h^2G(x, y)]_{x, y} \in \Omega_h^{(2)} \). Note that the minor \([h^2G(x, y)]_{x, y} \in \Omega_h^{(2)} \) is symmetric, while \( h^2G(x, y) = 0 \) for \( x \in \Omega_h^{(2)} \). For \( B \), we take the matrix \([h^2G(x, y)] \) whose eigenvalues are \([\lambda_i^{(n)}]^{-1} \). Thus, we must estimate \( ||h^2(G_h - G_i)||_2 \). However, for any matrix,

\[
||M||_2 \leq [\rho(MM^T)]^{1/2} \leq ||MM^T||_1^{1/2},
\]

where \( ||\cdot||_1 \) is the maximum of the absolute row sums of the matrix. This is a consequence of the Gerschgorin circle theorem (see, e.g., [7, p. 146]). Thus, we need to estimate

\[
(7.1) \quad h^4 \max_{x \in \Omega_h^{(2)}} \sum_{y \in \Omega_h^{(2)}} |[G_h(x, z) - G_i(x, z)][G_i(y, z) - G_i(y, z)]|.
\]

Let \( x_0 \) be the point where the max is attained and put

\[
\sigma(y) = \text{sgn}\ \sum_{x \in \Omega_h^{(2)}} |[G_h(x_0, z) - G_i(x_0, z)][G_i(y, z) - G_i(y, z)]|.
\]

Then, let

\[
W(x) = h^4 \sum_{y, z \in \Omega_h^{(2)}} |[G_h(x, z) - G_i(x, z)][G_i(y, z) - G_i(y, z)]\sigma(y)
\]

in (4.9). Then, (7.1) is bounded by

\[
(7.2) \quad Ch^4 \max_{x \in \Omega_h^{(2)}} \sum_{y \in \Omega_h^{(2)}} |G_h(y, z) - G_i(y, z)| + Ch^4 \max_{x \in \Omega_h^{(2)}} \sum_{y \in \Omega_h^{(2)}} |G_h(x, z) - G_i(x, z)|.
\]

Now,

\[
h^2 \sum_{y \in \Omega_h^{(2)}} |G_h(y, z) - G_i(y, z)| \leq C \max_{y, z \in \Omega_h^{(2)}} [||G_h(y, z)|| + ||G_i(y, z)||] \leq C|\log h|,
\]

by (4.11), (5.15) and (5.20). Using this in (7.2) and also (4.10) and (5.20), we have (7.2) bounded by \( Ch|\log h| \), which tends to zero as \( h \) tends to zero. Thus, the radii of the discs in Theorem 7.1 tend to zero as \( h \) does. Since the \( \mu_i^{(n)} \) tend to the \( \lambda_i^{(n)} \), which have no finite accumulation point, the disc associated with \([\mu_i^{(n)}]^{-1} \) for any
fixed \(n\) eventually becomes disjoint from the remaining discs. Consequently, for any fixed \(n\) and \(\epsilon > 0\), there is \(h\) sufficiently small that

\[
|\lambda_h^{(n)} - \lambda^{(n)}| < \epsilon.
\]

8. Main Theorem. We are now ready to state and prove our main theorem:

**Theorem 8.1.** Let \(\lambda^{(n)}\) be the \(n\)th eigenvalue of (1.1), let \(\lambda_h^{(n)}\) be the \(n\)th eigenvalue of (2.6) with associated eigenvector \(U_h^{(n)}\). For each \(n = 1, 2, \ldots\), there are constants \(C_n, h_n\) such that for \(h < h_n\)

\[
|\lambda_h^{(n)} - \lambda^{(n)}| < C_n h^4,
\]

and there is an eigenfunction \(u^{(n)}\) associated with \(\lambda^{(n)}\) such that

\[
\max_{\Omega_h} |U_h^{(n)} - u^{(n)}| < C_n h^4.
\]

**Proof.** With the machinery generated in the previous sections, our proof will have exactly the form of the proof of the corresponding Theorem 5.1 of [6]. For this reason, we only sketch the proof.

By (7.3)

\[
|\lambda_h^{(n)}| \leq C_n.
\]

By (5.11), (2.6) is equivalent to

\[
U_h^{(n)}(x) = \lambda_h^{(n)} h^2 \sum_{y \in \Omega_h} G(x, y) U_h^{(n)}(y), \quad x \in \Omega_h.
\]

Let us use the notations

\[
\langle U, V \rangle_h = h^2 \sum_{y \in \Omega_h} U(y) \overline{V(y)}, \quad ||U||_h = \langle U, U \rangle_h^{1/2},
\]

\[
\langle U, V \rangle_h' = h^2 \sum_{y \in \Omega_h} U(y) \overline{W(y)}, \quad ||U||_h' = \langle U, U \rangle_h'^{1/2}.\]

If \(U_h^{(n)}\) is normalized by requiring \(||U_h^{(n)}||_h = 1\), then (8.4), (8.3), the Schwarz inequality, and (5.16) show

\[
\max_{\Omega_h} |U_h^{(n)}| \leq C_n.
\]

From (8.4), (8.5) and (5.17), we see that for \(|x - \partial \Omega| \leq Ch\)

\[
|U_h^{(n)}(x)| \leq C_n h.
\]

Let us suppose that \(\lambda^{(n)} = \lambda^{(n+1)} = \cdots = \lambda^{(n+m)}\) is an eigenvalue of multiplicity \(m + 1\). Since \(\Delta_h\) restricted to \(\Omega_h\) is symmetric, the eigenvectors \(V_h^{(n)}\) of (6.1) are a complete orthonormal basis on \(\Omega_h\):

\[
\langle V_h^{(i)}, V_h^{(j)} \rangle_h = \delta(i, j).
\]

If we set

\[
P_h^{(i)} = \sum_{j=n}^{n+m} \langle U_h^{(i)}, V_h^{(j)} \rangle_h V_h^{(j)}, \quad i = n, \ldots, n + m,
\]

then
(8.7) \[ \| U_h^{(i)} - \tilde{V}_h^{(i)} \|_h^2 \leq C_h, \quad i = n, \ldots, n + m. \]

This follows from Parseval's identity:
\[
\| U_h^{(i)} \|_h^2 = \langle U_h^{(i)}, V_h^{(i)} \rangle_h + \sum_{j=n, \ldots, n+m} |\langle U_h^{(i)}, V_h^{(j)} \rangle_h|^2
\]
\[
= \langle U_h^{(i)}, V_h^{(i)} \rangle_h + \sum_{j=n, \ldots, n+m} \left| \frac{\mu_h^{(i)}}{\mu_h^{(i)} - \lambda_h^{(i)}} \langle H_h^{(i)}, V_h^{(j)} \rangle_h \right|^2,
\]
where \( H_h^{(i)} \) is uniquely defined by
\[
\Delta_h H_h^{(i)}(x) = 0, \quad x \in \Omega', \quad H_h^{(i)}(x) = U_h^{(i)}(x), \quad x \in \Omega^{(2)} \cup \Omega^{(3)}.
\]

It follows from our hard-won inequality (5.21) that
\[
\max_{\Omega_h} |H_h^{(i)}| \leq \max_{\Omega_h^{(x)} \cup \Omega_h^{(x)}} |U_h^{(i)}| \leq C_h,
\]
by (8.6), and so
\[
\| U_h^{(i)} - \tilde{V}_h^{(i)} \|_h^2 = \| U_h^{(i)} \|_h^2 - \langle U_h^{(i)}, \tilde{V}_h^{(i)} \rangle_h \leq C_h h^2.
\]

In a very similar manner, we show that if
\[
\tilde{V}_h^{(i)} = \sum_{i=n}^{n+m} \langle U_h^{(i)}, V_h^{(i)} \rangle_h V_h^{(i)}, \quad i = n, \ldots, n + m,
\]
then
\[
(8.8) \quad \| U_h^{(i)} - \tilde{V}_h^{(i)} \|_h \leq C_h, \quad i = n, \ldots, n + m.
\]

From (8.8), we can conclude that the \((m+1) \times (m+1)\) matrix \([\langle u_h^{(i)}, V_h^{(j)} \rangle_h], i, j = n, \ldots, n + m\), is nonsingular. In particular then, there are eigenvectors
\[
u_h^{(i)} = \sum_{j=n}^{n+m} a_{ij}(h) u_h^{(i)}, \quad i = n, \ldots, n + m,
\]
in the eigenmanifold associated with \(\lambda^{(n)}\) such that
\[
\langle u_h^{(i)}, V_h^{(j)} \rangle_h = \langle U_h^{(i)}, V_h^{(j)} \rangle_h, \quad i, j = n, \ldots, n + m.
\]

Moreover, the coefficients \(a_{ij}(h)\) are bounded independently of \(h\).

Then, it follows from (8.9) and Parseval's identity that
\[
\| U_h^{(i)} - u_h^{(i)} \|_h^2 = h^2 \sum_{\Omega_h^{(x)} \cup \Omega_h^{(x)}} |U_h^{(i)} - u_h^{(i)}|^2 + \sum_{i=n, \ldots, n+m} |\langle U_h^{(i)} - u_h^{(i)}, V_h^{(i)} \rangle_h|^2
\]
\[
= h^2 \sum_{\Omega_h^{(x)} \cup \Omega_h^{(x)}} |U_h^{(i)} - u_h^{(i)}|^2
\]
\[
+ \sum_{i=n, \ldots, n+m} \left| \frac{\mu_h^{(i)}}{\mu_h^{(i)} - \lambda_h^{(i)}} \langle H_h^{(i)}, V_h^{(i)} \rangle_h - \frac{\mu_h^{(i)}}{\mu_h^{(i)} - \lambda_h^{(i)}} \langle F_h^{(i)}, V_h^{(i)} \rangle_h \right|^2,
\]
where \(F_h^{(i)}\) is defined by
\[
\Delta_h F_h^{(i)}(x) = 0, \quad x \in \Omega', \quad F_h^{(i)}(x) = u_h^{(i)}(x), \quad x \in \Omega^{(2)} \cup \Omega^{(3)}.
\]

Since \(|u_h^{(i)}(x)| \leq C_h\) for \(|x - \partial\Omega| \leq C_h\), we see that
\[
(8.10) \quad \| U_h^{(i)} - u_h^{(i)} \|_h \leq C_h.
\]
From (8.10), we also have
\begin{equation}
|\langle U_h^{(i)}, u_h^{(i)} \rangle_h| \geq 1 - C_i h^2. \tag{8.11}
\end{equation}

Inequality (8.11) is the key inequality needed to prove the first half of Theorem 8.1, for now
\begin{equation}
(\lambda_h^{(i)} - \lambda^{(i)}) (U_h^{(i)}, u_h^{(i)}) = \langle U_h^{(i)}, \Delta u_h^{(i)} - \Delta_h^{*} u_h^{(i)} \rangle_h \tag{8.12}
\end{equation}

obtained by adding and subtracting terms. We have used the notations
\[
\Delta_h u_h^{(i)} = \Delta u_h^{(i)} - \Delta_h^{*} u_h^{(i)}
\]
for the truncation error, and $\Delta_h^{*}$ for the adjoint of $\Delta_h$ defined by
\[
\Delta_h^{*} V(x) = \sum_{y \in \Omega_h} l_h(y, x) V(y).
\]

Recall by (2.6) and our smoothness assumption on $u^{(i)}$ that
\[
|\tau_h u_h^{(i)}| \leq C_i h^4, \quad \text{on } \Omega_h,
\]
\[
\leq C_i h^2, \quad \text{on } \Omega_h^{(2)} \cup \Omega_h^{(3)}.
\]

However, on $\Omega_h^{(2)} \cup \Omega_h^{(3)}$ both $U^{(i)}$ and $u_h^{(i)}$ are bounded by $C_i h$, while the number of points in $\Omega_h^{(2)} \cup \Omega_h^{(3)}$ is only proportional to $h^{-1}$. From these considerations, we see that the first three terms on the right side of (8.12) are bounded by $C_i h^4$. As the remaining term,
\[
\Delta_h u_h^{(i)}(x) - \Delta_h^{*} u_h^{(i)}(x)
\]
vanishes for $x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}$, and is bounded by
\[
C_i h^{-2} \max_{\Omega_h \cup \Omega_h^{(2)} \cup \Omega_h^{(3)}} |u_h^{(i)}| \leq C_i h^{-1}
\]
for $x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}$. Again noting that the number of points in $\Omega_h^{(2)} \cup \Omega_h^{(2)} \cup \Omega_h^{(3)}$ is only proportional to $h^{-1}$, the last term on the right of (8.12) is bounded by
\[
C_i \max_{\Omega_h^{(2)} \cup \Omega_h^{(3)}} |U_h^{(i)} - u_h^{(i)}| .
\]

Thus, using (8.11) we have the inequality
\begin{equation}
|\lambda_h^{(i)} - \lambda^{(i)}| \leq C_i \left[ \max_{\Omega_h^{(2)} \cup \Omega_h^{(3)}} |U_h^{(i)} - u_h^{(i)}| + h^4 \right]. \tag{8.13}
\end{equation}

We next employ the discrete Green's function to write
\begin{equation}
U_h^{(i)}(x) - u_h^{(i)}(x) = h^2 \sum_{y \in \Omega_h} G_h(x, y) \Delta_h[u_h^{(i)}(y) - U_h^{(i)}(y)] \tag{8.14}
\end{equation}

Using inequalities (5.13) and (5.14), we see that the first term on the right of (8.14) is bounded by $C_i h^4$.

By (5.14) and (8.5) the last term on the right is bounded by
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The last term bounded by \( C, h \), (5.17) shows the last term bounded by \( C, h \), \( |\lambda^{(i)} - \lambda^{(i)}| \), or if \( |x - \partial \Omega| \leq Ch \), (5.17) shows the last term bounded by \( C, h \), \( |\lambda^{(i)} - \lambda^{(i)}| \). Using (8.3), (5.16) and Schwarz's inequality bound the middle term on the right by \( ||U^{(i)} - u^{(i)}||_h \), or, if \( |x - \partial \Omega| \leq Ch \), (5.17) bounds it by \( C, h \max_{\Omega} |U^{(i)} - u^{(i)}| \). In summary,

\[
\max_{\Omega} |U^{(i)} - u^{(i)}| \leq C_i \left[ \max_{\Omega} |U^{(i)} - u^{(i)}| + |\lambda^{(i)} - \lambda^{(i)}| + h^4 \right].
\]

Finally, we use Parseval's identity and (8.9) to conclude that

\[
||U^{(i)} - u^{(i)}||^2_h = \sum_{f \in \Omega^{(i)}} |U^{(i)}(f) - u^{(i)}(f)|^2 + \sum_{f \neq f_1, \ldots, f_{n+m}} |(U^{(i)} - u^{(i)}, V^{(i)}_h)|^2,
\]

and by a straightforward computation

\[
(\lambda^{(i)} - \lambda^{(i)})(U^{(i)} - u^{(i)}, V^{(i)}_h)_h = (\kappa^{(i)} - \lambda^{(i)})U^{(i)}_h - \eta u^{(i)}_h + \tilde{H}_h^{(i)}, V^{(i)}_h)_h,
\]

where \( \tilde{H}_h^{(i)} \) is defined by

\[
\Delta_h \tilde{H}_h^{(i)}(x) = 0, \quad x \in \Omega^{(i)}, \quad \tilde{H}_h^{(i)}(x) = U^{(i)}_h(x) - u^{(i)}_h(x), \quad x \in \Omega^{(i)} \cup \Omega^{(3)}.
\]

It follows that

\[
||U^{(i)} - u^{(i)}||_h \leq C_i \left[ \max_{\Omega^{(i)} \cup \Omega^{(3)}} |U^{(i)} - u^{(i)}| + |\lambda^{(i)} - \lambda^{(i)}| + h^4 \right].
\]

Combining (8.13), (8.15), (8.16), and (8.17) yields the proof of Theorem 8.1.

Let us observe some simple consequences of Theorem 8.1. Since the \( \lambda^{(i)} \) are real, we have

\[
|\text{Re} \lambda^{(i)} - \lambda^{(i)}| \leq Ch^4.
\]

Also, when \( \lambda^{(i)} \) is simple, \( \lambda^{(i)} \) will be real for \( h \) sufficiently small. This is because the matrix \( [\lambda^{(i)}]_{x \in \Omega^{(i)}} \) is real. Thus, if \( \lambda^{(i)} \) were complex, its conjugate \( [\lambda^{(i)}] \) would also be a distinct eigenvalue of \( \Delta_h \) converging to \( \lambda^{(i)} \). But this is impossible, since \( [\lambda^{(i)}] \) must converge to some \( \lambda^{(i)} = \lambda^{(i)} \).

We normalized \( U^{(i)}_h \) by requiring \( ||U^{(i)}||_h = 1 \). This determines \( U^{(i)}_h \) only up to a multiplicative constant of modulus 1. If we specify this constant by requiring that \( (U^{(i)}_h, V^{(i)}_h)_h \geq 0 \), then when \( \lambda^{(i)} \) is simple, \( u^{(i)}(x) \) is a real multiple of \( u^{(i)}(x) \), as can be seen from (8.9).

Theorem 8.1 shows that \( U^{(i)}_h \) approximates to \( O(h^4) \) an eigenfunction \( u^{(i)}(x) \) which depends on \( h \). Properly normalized, however, \( U^{(i)}_h \) will approximate to \( O(h^4) \) an eigenfunction \( u^{(i)}(x) \) such that \( \int_\Omega |u^{(i)}(x)|^2 dx = 1 \), independently of \( h \). In particular, when \( \lambda^{(i)} \) is simple, \( U^{(i)}_h \) will approximate the unique normalized eigenfunction \( u^{(i)}(x) \). This normalization is

\[
h^2 \sum_{x \in \Omega} \alpha_h(y) |U^{(i)}_h(y)|^2 = 1,
\]

where \( \alpha_h \) is given in the appendix of [6]. For a proof, see [6, Corollary 6.2].

9. Forced Vibration Problems. Let us remark that all of the results of the previous sections hold for the problem

\[
\Delta u(x) + (q(x) + \lambda)u(x) = 0, \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial \Omega,
\]
where \( q \) is nonpositive and smooth on \( \Omega \), and for the discrete Green's function \( G_k \) defined by

\[
(\Delta_{h,z} + q(z))G_k(x, y) = -h^{-2} \delta(x, y), \quad x, y \in \Omega_k.
\]

The proofs require only that the additional term \( q \) be carried along throughout. We make this remark because we next wish to consider the problem

\[
(\Delta_{h,z} + r(z))u(x) = F(x), \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial \Omega,
\]

for \( F \) and \( r \) given smooth functions on \( \Omega \). Problem (9.3) is a forced vibration problem and an \( O(h^2) \) analogue of it was studied by Bramble in [1].

Let us rewrite (9.3) in the form

\[
\Delta_{h,k}u(x) + q(x)u(x) + \left( \sup_{g} r \right)u(x) = F(x), \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial \Omega,
\]

where \( q(x) = r(x) - \sup_{g} r \leq 0 \) on \( \Omega \). A unique solution \( u \) of (9.3) or (9.4) exists if and only if \( \sup_{g} r \) is not an eigenvalue of the operator \( \Delta + q \). Now, we consider the difference approximation

\[
\Delta_{h,k}U_h(x) + r(x)U_h(x) = F(x), \quad x \in \Omega_k,
\]

where \( \Delta_{h,k} \) is the difference operator defined in Section 2. We prove:

**Theorem 9.1.** If (9.3) has a unique solution \( u \in C^0(\Omega) \), there are constants \( C, h_0 \) such that for \( h < h_0 \), (9.5) has a unique solution \( U_h \) for which

\[
\max_{\Omega_h} |U_h - u| < C h^4.
\]

**Proof.** Let \( G_k \) be the discrete Green's function defined in (9.2). Then, for \( x \in \Omega_k \),

\[
|U_h(x) - u(x)| = h^2 \sum_{y \in \Omega_k} G_k(x, y)\left| \Delta_{h,k}u(y) + q(y)u(y) - \Delta_{h,k}U_h(y) - q(y)U_h(y) \right|
\]

\[
\leq \sup_{y \in \Omega_k} |q| h^2 \sum_{y \in \Omega_k} |G_k(x, y)| \left| U_h(y) - u(y) \right| + h^2 \sum_{y \in \Omega_k} |G_k(x, y)| \left| r(x)u(y) \right|.
\]

Therefore, using (5.13) and (5.14) for \( G_k \) of (9.2) and (2.5),

\[
|U_h(x) - u(x)| \leq C \left[ h^2 \sum_{y \in \Omega_k} |G_k(x, y)| \left| U_h(y) - u(y) \right| + h^4 \right].
\]

Employing (5.17), this yields

\[
\max_{\Omega_h} \left| U_h - u \right| \leq C \left[ h \max_{\Omega_h} \left| U_h - u \right| + h^4 \right],
\]

while (5.16) and Schwarz's inequality yield

\[
\max_{\Omega_h} \left| U_h - u \right| \leq C \left[ \left| U_h - u \right| \| u \|_1 + h^4 \right].
\]

From (9.7) and (9.8), we see

\[
\left| U_h - u \right| \| u \|_1 \leq C \left[ \| U_h - u \|_1 + h^4 \right].
\]
which implies

\[(9.9) \quad ||U_h - u||_h \leq C(||U_h - u||^*_h + h^4).\]

Finally, we complete the proof by using Parseval’s identity to estimate

\[(9.10) \quad ||U_h - u||^*_h = \left[ \sum |\langle U_h - u, V_h^{(i)} \rangle_h^*|^2 \right]^{1/2},\]

where \(V_h^{(i)}\) is the eigenvector associated with \(\mu_h^{(i)}\) in the symmetric problem

\[\Delta_h V_h^{(i)}(x) + (q(x) + \mu_h^{(i)} V_h^{(i)}(x) = 0, \quad x \in \Omega', \quad V_h^{(i)}(x) = 0, \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}\].

Define \(H_h\) by

\[\Delta_h H_h(x) + q(x)H_h(x) = 0, \quad x \in \Omega', \quad H_h(x) = U_h(x) - u(x), \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}\].

From (5.21), we have

\[\max \{H_h\} \leq C \max \{|U_h - u|\},\]

or, employing (9.7), (9.8), (9.9),

\[(9.11) \quad \max \{H_h\} \leq C[h ||U_h - u||^*_h + h^4].\]

Then, we have

\[
\mu_h^{(i)}\langle U_h - u, V_h^{(i)} \rangle_h^* = \langle H_h + u - U_h, (\Delta_h + q)V_h^{(i)} \rangle_h + \mu_h^{(i)}\langle H_h, V_h^{(i)} \rangle_h^*
\]

\[= \langle (\Delta_h + q)(H_h + u - U_h), V_h^{(i)} \rangle_h + \mu_h^{(i)}\langle H_h, V_h^{(i)} \rangle_h^*
\]

\[= (\sup r)(U_h - u, V_h^{(i)} \rangle_h^* - \langle r, V_h^{(i)} \rangle_h^* + \mu_h^{(i)}\langle H_h, V_h^{(i)} \rangle_h^*\].

Now, since \(\text{sup } r\) is not an eigenvalue \(\lambda^{(i)}\) of \(\Delta + q\) and \(\mu_h^{(i)} \to \lambda^{(i)}\) as \(h \to 0\), there are constants \(C, h_0\) such that for \(h < h_0\),

\[\max \{|\mu_h^{(i)} - \text{sup } r|^{-1} < C, \quad \max \{|\mu_h^{(i)}/|\mu_h^{(i)} - \text{sup } r| < C,\]

and so

\[|\langle U_h - u, V_h^{(i)} \rangle_h^*| \leq C[|\langle r, V_h^{(i)} \rangle_h^*| + |\langle H_h, V_h^{(i)} \rangle_h^*|].\]

Using this in (9.10), we see that

\[||U_h - u||^*_h \leq C(||r, u||_h^* + ||H_h||_h^*) \leq C[h^4 + h ||U_h - u||^*_h],\]

by (9.11), from which it follows that

\[||U_h - u||^*_h \leq C h^4,\]

completing the proof.

Let us remark that by employing the results of [6], the above technique of proof will show that a unique solution of the forced vibration problem:

\[(9.12) \quad \sum \frac{\partial^2 u(x)}{\partial \alpha \partial \beta} \left( u_i(\alpha) \frac{\partial u(\alpha)}{\partial \alpha} \right) ; \quad r(\alpha)u(\alpha) = I(\alpha), \quad x \in \Omega',
\]

\[u(\alpha) = 0, \quad x \in \partial \Omega,\]
can be approximated to $O(h^2)$ by using the symmetric difference scheme given in [6] at the beginning of Section 7.

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