Exponential Chebyshev Approximation on Finite Subsets of [0, 1]

By Bernard H. Rosman

Abstract. In this note the convergence of best exponential Chebyshev approximation on finite subsets of [0, 1] to a best approximation on the interval is proved when the function to be approximated is continuous and when the union of the finite subsets is dense in [0, 1].

1. Introduction. In this note we study the convergence of best exponential Chebyshev approximation on finite subsets of [0, 1] to a best approximation on the interval. This problem has been considered for linear approximation [1] and, recently, for generalized rational approximation in [2].

Let $X_r$ be a set of $r$ distinct points in [0, 1], containing the endpoints (a common computational situation). We assume that the sequence of subsets $\{X_r\}$ fills up the interval in the sense that, given $x \in [0, 1]$, there is an $x_r \in X_r$ such that $\{x_r\} \to x$.

Following Rice [3, Chapter 8], we approximate $f \in C([0, 1])$ by exponential functions of the form

$$ E(A, x) = \sum_{i=1}^{k} \left( \sum_{j=0}^{m_i} p_{i,j} x^j \right) e^{i t_x}, $$

where $|p_i| < \infty$, $|t_i| < \infty$ and $\sum_{i=1}^{k} (m_i + 1) \leq n$, $n$ a fixed positive integer.

For each set $X_r$, we define the usual seminorm on $C([0, 1])$, corresponding to $X_r$, by

$$ ||f||_{X_r} = \sup_{x \in X_r} |f(x)| $$

for all $f$ in $C([0, 1])$. We denote by $E(A_r, x)$ and $E_r(x)$ the best approximation to $f(x)$ on $X_r$, i.e. the best approximation to $f$ with respect to the seminorm corresponding to $X_r$. Norm signs without subscripts denote the usual Chebyshev norm.

It is known [3] that best approximation need not exist on finite point sets. However, we assume existence, a reasonable assumption in many computational situations. Moreover, Rice solves a special case of this problem through the use of pseudo functions [3, pp. 65–69]. An extension of this technique to handle general exponential approximation is under investigation.

2. The Convergence Theorem. The main result of this note is

THEOREM 1. Let $E^*$ be a best exponential approximation to $f$ on [0, 1]. Then

$$ ||f - E_r||_{X_r} \to ||f - E^*||. $$

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575
Proof. First, suppose that \(||E_r|||\) is a bounded sequence. Then by a theorem in Rice [3, Chapter 8], \({E_r(x)}\), or a subsequence thereof, converges pointwise to a possibly discontinuous function

\[
E(x) = E_0(x), \quad 0 < x < 1,
\]

\[
= e_0, \quad x = 0,
\]

\[
= e_1, \quad x = 1,
\]

where \(E_0(x)\) is an exponential of form (1). We claim that \(E_0\) is a best approximation to \(f\) on \([0, 1]\). For if not, there exists a point \(x\) such that \(|f(x) - E_0(x)| > ||f - E^*||\). By continuity, we may assume that \(x \in [\delta, 1 - \delta]\) where \(\delta > 0\) is sufficiently small. Let \(\{x_r\} \rightarrow x\) with \(x_r \in X_r, \epsilon > 0\), and let \(y_1, \ldots, y_n\) be in \([\delta, 1 - \delta]\). By the definition of varisolvence, there is a \(\delta(\epsilon)\) such that the inequality \(|E_r(y) - E_0(y)| < \delta(\epsilon)\) implies that there is a function \(E(A, y)\) satisfying \(E(A, y) = Y_r\) and \(|E(A, y) - E_0(y)| < \epsilon\). But since \(E_r \rightarrow E_0\) pointwise on \([\delta, 1 - \delta]\), for \(r\) large enough, \(|E_r(y) - E_0(y)| < \delta(\epsilon)\). Hence, there are functions \(\{E_r(x, y)\}\) such that \(E_r(x, y) = E_0(x, y)\). But since these interpolating functions are unique (\(m\) is the degree of solvence), \(E(A, y) = E_r(y)\) for all \(y\). Therefore, \(|E_r(y) - E_0(y)| < \epsilon\) for \(y\) in \([\delta, 1 - \delta]\). Therefore, by a standard inequality, \(|f(x, y) - E_r(x, y)| < ||f - E^*||\). This contradicts the fact that \(E_r\) is a best approximation on \(X_r\), and hence \(E_0\) is a best approximation.

Suppose now that \(||E_r|||\) is an unbounded sequence. Following Dunham [2], define \(B_r(x) = E_r(x)/||E_r||\). Then \(||B_r|||\) is a bounded sequence and, by the aforementioned result of Rice, we may assume that \(\{B_r\}\) converges pointwise to a function of the form

\[
B(x) = B_0(x), \quad 0 < x < 1,
\]

\[
= b_0, \quad x = 0,
\]

\[
= b_1, \quad x = 1,
\]

where \(B_0(x)\) is an exponential of form (1). Now, assume that \(b_0 = B_0(0)\) and \(b_1 = B_0(1)\). The \(B(x)\) is an exponential and, using varisolvence as before, it follows that \(\{B_r\}\) converges uniformly to \(B\). Since \(||B_r||| = 1\), there exists \(y \in [0, 1]\) and a neighborhood \(N\) of \(y\) such that \(m = \inf_{x \in N} B(x) > 0\) and \(B(x) > m/2\) for \(r\) sufficiently large. Hence, \(\inf_{x \in N} E_r(x) \rightarrow \infty\) as \(r \rightarrow \infty\) and, for large enough \(r\), there exists \(x_r \in X_r\) such that \(E_r(x_r) > 2||f||\). This contradicts \(E_r\) being a best approximation to \(f\) on \(X_r\), since then

\[
||f - E_r||_{X_r} > ||f|| \geq ||f||_{X_r} = ||f - 0||_{X_r}.
\]

It remains to consider the case where \(B(x)\) has an endpoint discontinuity. Assume without loss of generality that \(B(x) = B_0(x), 0 < x \leq 1, \text{ and } b_0 > B_0(0)\). If \(B_0 \neq 0\), there exists \(y \in [\delta, 1 - \delta], \delta > 0\), such that \(B_0(y) \neq 0\) and \(\{B_0(y)\} \rightarrow B_0(y)\). Using the varisolvence of \(B_0\) on this interval, we again contradict \(E_r\), being best on \(X_r\). If \(B_0 = 0\), we reach the same contradiction on \(E_r\), since then \(\{E_r(0)\}\) is unbounded. This concludes the proof.

If we cannot assume that the endpoints are in \(X_r\), the proof goes through without change except for the last case. If \(B_0 = 0\) and if \(0 \in X_r\), for only a finite set of \(r\) values
with $b > 0$, it follows from the properties of exponentials and the boundedness of $||E_r||_X$, that $\{E_r(x)\}$ is bounded on $[\delta, 1 - \delta]$, $\delta > 0$. Hence, $\{E_r(x)\}$ or a subsequence thereof converges pointwise to $E_0(x)$, an exponential, on $(0, 1)$. From this reasoning, as before using varisolvence, $E_0$ is a best approximation to $f$ on $[0, 1]$.

From the proof of Theorem 1 we have

**Corollary.** The sequence $\{E_r(x)\}$ has a subsequence which converges pointwise except possibly at the endpoints to a best exponential approximation, $E_0(x)$, to $f$ on $[0, 1]$. If the subsequence converges to $E_0(0)$ and $E_0(1)$ also, then the convergence is uniform.

Department of Mathematics
Boston University
Boston, Massachusetts 02215

