Exponential Chebyshev Approximation on Finite Subsets of \([0, 1]\)

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Abstract. In this note the convergence of best exponential Chebyshev approximation on finite subsets of \([0, 1]\) to a best approximation on the interval is proved when the function to be approximated is continuous and when the union of the finite subsets is dense in \([0, 1]\).

1. Introduction. In this note we study the convergence of best exponential Chebyshev approximation on finite subsets of \([0, 1]\) to a best approximation on the interval. This problem has been considered for linear approximation [1] and, recently, for generalized rational approximation in [2].

Let \(X_r\) be a set of \(r\) distinct points in \([0, 1]\), containing the endpoints (a common computational situation). We assume that the sequence of subsets \(\{X_r\}\) fills up the interval in the sense that, given \(x \in [0, 1]\), there is an \(x_r \in X_r\) such that \(\{x_r\} \to x\).

Following Rice [3, Chapter 8], we approximate \(f \in C([0, 1])\) by exponential functions of the form

\[
E(A, x) = \sum_{i=1}^{k} \left( \sum_{j=0}^{m_i} p_{i,j} \alpha^j \right) e^{i \alpha x},
\]

where \(|p_{i,j}| < \infty\), \(|\alpha| < \infty\) and \(\sum_{i=1}^{k} (m_i + 1) \leq n\), \(n\) a fixed positive integer.

For each set \(X_r\), we define the usual seminorm on \(C([0, 1])\), corresponding to \(X_r\), by

\[
||f||_{X_r} = \sup_{x \in X_r} |f(x)|
\]

for all \(f \in C([0, 1])\). We denote by \(E(A_r, x)\) and \(E_r(x)\) the best approximation to \(f(x)\) on \(X_r\), i.e. the best approximation to \(f\) with respect to the seminorm corresponding to \(X_r\). Norm signs without subscripts denote the usual Chebyshev norm.

It is known [3] that best approximation need not exist on finite point sets. However, we assume existence, a reasonable assumption in many computational situations. Moreover, Rice solves a special case of this problem through the use of pseudo functions [3, pp. 65–69]. An extension of this technique to handle general exponential approximation is under investigation.

2. The Convergence Theorem. The main result of this note is

**Theorem 1.** Let \(E^*\) be a best exponential approximation to \(f\) on \([0, 1]\). Then

\[
||f - E_r||_{X_r} \to ||f - E^*||.
\]

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Proof. First, suppose that \(||E_\nu||\) is a bounded sequence. Then by a theorem in Rice [3, Chapter 8], \(|E_\nu(x)|\), or a subsequence thereof, converges pointwise to a possibly discontinuous function

\[
E(x) = E_0(x), \quad 0 < x < 1, \\
= 0, \quad x = 0, \\
= 1, \quad x = 1,
\]

where \(E_0(x)\) is an exponential of form (1). We claim that \(E_0\) is a best approximation to \(f\) on \([0, 1]\). For if not, there exists a point \(x\) such that \(|f(x) - E_0(x)| > ||f - E^*||\). By continuity, we may assume that \(x \in [\delta, 1 - \delta]\) where \(\delta > 0\) is sufficiently small. Let \(\{x_r\} \rightarrow x\) with \(x_r \in X_r, \epsilon > 0\), and let \(y_1, \ldots, y_n\) be in \([\delta, 1 - \delta]\). By the definition of varisolvence, there is a \(\nu(\epsilon)\) such that the inequality \(|y_i - E_0(y_i)| < \nu(\epsilon)\) implies that there is a function \(E(A, y)\) satisfying \(E(A, y_i) = Y_i\) and \(|E(A, y) - E_0(y)| < \epsilon\). Since \(E_\nu(x)\) converges pointwise on \([\delta, 1 - \delta]\), for \(r\) large enough, \(|E_\nu(y) - E_0(y)| < \nu(\epsilon)\). Hence, there are functions \(\{E(A_r, y)\}\) such that \(E(A_r, y) = E_r(y)\). But since these interpolating functions are unique (\(m\) is the degree of solvence), \(E(A_r, y) = E_\nu(y)\) for all \(y\). Therefore, \(|E(y) - E_0(y)| < \epsilon\) for \(y\) in \([\delta, 1 - \delta]\). Therefore, by a standard inequality, \(|f(x_r) - E_\nu(x_r)| < ||f - E^*||\). But since \(E_\nu = E_\nu(x)\) for \(|f - E^*||\), this contradicts the fact that \(E_\nu\) is a best approximation on \(X_r\), and hence \(E_0\) is a best approximation.

Suppose now that \(||E_\nu||\) is an unbounded sequence. Following Dunham [2], define \(B_r(x) = E_\nu(x)/||E_\nu||\). Then \(||B_\nu||\) is a bounded sequence and, by the aforementioned result of Rice, we may assume that \(\{B_r(x)\}\) converges pointwise to a function of the form

\[
B(x) = B_0(x), \quad 0 < x < 1, \\
= 0, \quad x = 0, \\
= 1, \quad x = 1,
\]

where \(B_0(x)\) is an exponential of form (1). Now, assume that \(b_0 = B_0(0)\) and \(b_1 = B_0(1)\). The \(B(x)\) is an exponential and, using varisolvence as before, it follows that \(\{B_r\}\) converges uniformly to \(B\). Since \(||B||\) \(= 1\), there exists \(y \in [0, 1]\) and a neighborhood \(N\) of \(y\) such that \(m = \inf_{x \in N} B(x) > 0\) and \(B(x) > m/2\) for \(r\) sufficiently large. Hence, \(\inf_{x \in X} E_r(x) \rightarrow \infty\) as \(r \rightarrow \infty\) and, for large enough \(r\), there exists \(x_r \in X\), such that \(E_r(x_r) > 2||f||\). This contradicts \(E_r\) being a best approximation on \(X\), and hence \(E_0\) is a best approximation.

It remains to consider the case where \(B(x)\) has an endpoint discontinuity. Assume without loss of generality that \(B(x) = B_0(x), 0 < x \leq 1, \) and \(b_0 > B_0(0)\). If \(B_0 \neq 0\), there exists \(y \in [\delta, 1 - \delta], \delta > 0\), such that \(B_0(y) \neq 0\) and \(\{B_0(x)\} \rightarrow B_0(y)\). Using the varisolvence of \(B_0\) on this interval, we again contradict \(E_\nu\), being best on \(X_r\). If \(B_0 = 0\), we reach the same contradiction on \(E_\nu\) since then \(\{E_\nu(0)\}\) is unbounded. This concludes the proof.

If we cannot assume that the endpoints are in \(X_r\), the proof goes through without change except for the last case. If \(B_0 = 0\) and if \(0 \in X\), for only a finite set of \(r\) values
with \( b > 0 \), it follows from the properties of exponentials and the boundedness of 
\( ||E_r||_x \), that \( \{E_r(x)\} \) is bounded on \([\delta, 1 - \delta]\), \( \delta > 0 \). Hence, \( \{E_r(x)\} \) or a subsequence thereof converges pointwise to \( E_0(x) \), an exponential, on \((0, 1)\). From this reasoning, as before using varisolvence, \( E_0 \) is a best approximation to \( f \) on \([0, 1]\).

From the proof of Theorem 1 we have

**Corollary.** The sequence \( \{E_r(x)\} \) has a subsequence which converges pointwise except possibly at the endpoints to a best exponential approximation, \( E_0(x) \), to \( f \) on \([0,1]\). If the subsequence converges to \( E_0(0) \) and \( E_0(1) \) also, then the convergence is uniform.

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