A Useful Approximation to $e^{-t^2}$

By Richard Bellman, B. G. Kashef and R. Vasudevan*

Abstract. Using differential approximation, we obtain a remarkably accurate representation of $e^{-t^2}$ as a sum of three exponentials.

1. Introduction. The function $e^{-t^2}$ occurs in many important contexts in mathematics. In some of these, it is quite useful to replace it by an approximation of some type, such as, for example, a Padé approximation. In this note, we wish to exhibit a surprisingly good approximation as a sum of three exponentials. This is obtained using differential approximation, [1]. The approximation obtained here holds for $0 \leq t \leq 1$.

2. Differential Approximation. Given a function $k(t)$ for $0 \leq t \leq T$, we determine the coefficients $a_1, a_2, \ldots, a_N$ which minimize the quadratic expression

\[
J(a_i) = \int_0^T \left[ k^{(N)} + \sum_{i=1}^N a_i k^{(N-i)} \right]^2 dt,
\]

where $k^{(i)}$ denotes the $i$th derivative. We then expect that the solution of the linear differential equation

\[
u^{(N)} + a_1 \nu^{(N-1)} + \cdots + a_N \nu = 0,
\]

with suitable boundary conditions, will yield an approximation to $k(t)$. This is a question in stability theory.

The procedure is most useful when $N$ can be taken small. In this case, $N = 3$ and $5$ yield excellent results for $k(t) = e^{-t^2}$, as is demonstrated below.

3. Numerical Results. It turns out that good results are obtained by choosing as initial conditions in (2.2): $u^{(i)}(0) = k^{(i)}(0), i = 0, 1, \ldots, N - 1$. The coefficients $a_i$ are listed in the first column of Table 1.

For the case $N = 3$, the calculated values of $u, u', u'', \ldots$ agree to eight figures with the exact values $k, k', k'', \ldots$, respectively. The accuracy is even better for $N = 5$.

If we express the solution of the linear differential equation as a sum of exponentials, we obtain the expression

\[
u(t) = \sum_{i=1}^N b_i \exp(-\lambda_i t),
\]

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where \( b_i \) and \( \lambda_i \) can have complex values. These values are calculated and listed in Table 1. The numerical values of function \( u(t) \) of the above equation at different time intervals \( (0 \leq t \leq 1) \) are listed in Table 2. In the same table, the absolute errors are also shown.

### Table 1

<table>
<thead>
<tr>
<th>( N )</th>
<th>( a_i )</th>
<th>( b_i )</th>
<th>( \lambda_i )</th>
</tr>
</thead>
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<tr>
<td>2.7403</td>
<td>.7853</td>
<td>.9180</td>
<td></td>
</tr>
<tr>
<td>7.9511</td>
<td>.1074 + .1963</td>
<td>.9111 + .1 2.334</td>
<td></td>
</tr>
<tr>
<td>5.7636</td>
<td>.1074 - .1963</td>
<td>.9111 - .9111 - .1 2.334</td>
<td></td>
</tr>
<tr>
<td>4.7471</td>
<td>.6509</td>
<td>.9509</td>
<td></td>
</tr>
<tr>
<td>27.9415</td>
<td>.1795 + .2204</td>
<td>.9503 + .9503 + .1 2.866</td>
<td></td>
</tr>
<tr>
<td>62.5129</td>
<td>.1795 - .2204</td>
<td>.9503 - .9503 - .1 2.866</td>
<td></td>
</tr>
<tr>
<td>109.1101</td>
<td>-.0049 + .0163</td>
<td>.9473 + .9473 + .1 3.930</td>
<td></td>
</tr>
<tr>
<td>68.1498</td>
<td>-.0049 - .0163</td>
<td>.9473 - .9473 - .1 3.930</td>
<td></td>
</tr>
</tbody>
</table>

### Table 2

<table>
<thead>
<tr>
<th>Time</th>
<th>Calculated Value</th>
<th>Absolute Error</th>
<th>Calculated Value</th>
<th>Absolute Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N = 3 )</td>
<td>( N = 5 )</td>
<td>( N = 3 )</td>
<td>( N = 5 )</td>
<td></td>
</tr>
<tr>
<td>.1</td>
<td>.990020</td>
<td>.30 \times 10^{-4}</td>
<td>.990049</td>
<td>.4 \times 10^{-8}</td>
</tr>
<tr>
<td>.3</td>
<td>.913676</td>
<td>.255 \times 10^{-3}</td>
<td>.913931</td>
<td>.2 \times 10^{-6}</td>
</tr>
<tr>
<td>.5</td>
<td>.778679</td>
<td>.122 \times 10^{-3}</td>
<td>.778800</td>
<td>.2 \times 10^{-7}</td>
</tr>
<tr>
<td>.8</td>
<td>.527665</td>
<td>.372 \times 10^{-6}</td>
<td>.527292</td>
<td>.2 \times 10^{-6}</td>
</tr>
<tr>
<td>1.0</td>
<td>.367951</td>
<td>.72 \times 10^{-4}</td>
<td>.367879</td>
<td>.2 \times 10^{-6}</td>
</tr>
</tbody>
</table>

4. Discussion. If desired, we can improve the accuracy of the approximation by taking the values of \( u^{(i)}(0) \), the initial conditions, as parameters, \( u^{(i)}(0) = c_i \), and then, by determining these values by the minimization of the quadratic expression,

\[
J(c_i) = \int_0^T \left[ k(t) - \sum_{i=1}^N c_i u_i(t) \right]^2 dt,
\]

where \( u_1, \ldots, u_N \) are \( N \) linearly independent solutions of (2.2).

The integrals which arise are evaluated by using the differential equation (2.2) plus the auxiliary equations

\[
\frac{d v_{i,t}}{dt} = u_i u_t, \quad v_{i,t}(0) = 0, \quad \frac{d w_i}{dt} = u_i k, \quad w_i(0) = 0.
\]

Then,

\[
v_{i,t}(T) = \int_0^T u_i u_t dt, \quad w_i(T) = \int_0^T u_i k dt.
\]

The same technique can often be used in the determination of the coefficients \( a_i \)
when the function $k(t)$ satisfies a differential equation, linear or nonlinear. In this case, $k' = -t^2k, k(0) = 1.$

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