Miniaturized Tables of Bessel Functions. III*

By Yudell L. Luke

Abstract. After the manner of our previous studies, coefficients for the expansion of $J_\nu(z)$ and $Y_\nu(z)$ in double series of Chebyshev polynomials are presented. For $J_\nu(z)$, the ranges are (1) $0 < z \leq 8$, $0 \leq \nu \leq 4$, (2) $0 < z \leq 8$, $4 \leq \nu \leq 8$. For $J_\nu(z) + iY_\nu(z)$, the ranges are $z \geq 5$ and $0 \leq \nu \leq 1$. The coefficients are given with sufficient accuracy to enable the evaluation of the Bessel functions to at least 20 decimals.

1. Introduction. In previous studies [1], [2], we considered the expansion of two parameter functions in a double series of Chebyshev polynomials and developed coefficients for the evaluation of $K_\nu(z)$ and $I_\nu(z)$ over a large part of the real $z$ and $\nu$ lines. In the present paper, we give similar type coefficients for the evaluation of $J_\nu(z)$ and $Y_\nu(z)$.


\[
J_\nu(z) = z^\nu \sum_{k=0}^{\infty} A_k(\nu, \lambda) T_{2k}(z/\lambda),
\]

for $0 < z \leq \lambda$.

(2)

\[
A_k(\nu, \lambda) = G_k(\nu, \lambda)/2^k \Gamma(\nu + 1),
\]

(3)

\[
G_k(\nu, \lambda) = \frac{\epsilon_k(-)^k \lambda^{2k} \Gamma(\nu + 1)}{2^k k! \Gamma(k + \nu + 1)},\]

where

\[
\epsilon_k = \begin{cases} 1 & \text{if } k = 0, \\ 2 & \text{if } k > 0. \end{cases}
\]

It is readily shown that

\[
G_k(\nu, \lambda) = \frac{\epsilon_k(-)^k \lambda^{2k} \Gamma(\nu + 1)}{2^k k! \Gamma(k + \nu + 1)} \left[ 1 + O(k^{-1}) \right],
\]

and for $\nu$ and $\lambda$ fixed,
Thus, the expansion (1) converges and by letting $z \to 0$, we have the useful normalization relation

$$
\sum_{k=0}^{\infty} (-1)^k A_k(\nu, \lambda) = 1.
$$

Further, after the manner of the discussion presented in [3, Vol. 2, pp. 159-166], we can show that use of the recursion formula (4) in the backward direction is convergent. Thus, for a fixed $\lambda$, we can generate the coefficients $A_k(\nu, \lambda)$ for any given value of $\nu$. Then, following the discussion given in [1], we can develop coefficients $D_{r,k}(\lambda)$ such that

$$
A_k(\nu, \lambda) = \sum_{r=0}^{\infty} D_{r,k}(\lambda) T_r(s), \quad s \leq \nu \leq s + t.
$$

We remark that 20 decimal values of $A_k(\nu, \lambda)$ are given in [3, pp. 331, 332, 352-356] for $\lambda = 8$ and $\nu = 0, \pm \frac{1}{8}, \pm \frac{1}{4}, \pm \frac{3}{8}, \pm \frac{1}{2}, 1$. Coefficients for the evaluation of $Y_0(z)$ and $Y_1(z)$ for $0 < z \leq 8$ are also given in [3, pp. 331, 332].

We next present a descending type expansion in series of Chebyshev polynomials for the evaluation of $J_r(z)$ and $Y_r(z)$ in the vicinity of $z = +\infty$. Now,

$$
H_r^{(1)}(z) = -\frac{2i}{\pi} e^{-iz/2} K_r(ze^{-i\pi/2}),
$$

and from [1], we have

$$
K_r(z) = (\pi/2z)^{1/2} e^{-z} \sum_{k=0}^{\infty} G_k(\nu, \lambda) T_k(\nu/z), \lambda \text{ fixed}, \nu/z \leq 1, |\operatorname{arg} z| < 3\pi/2.
$$

The recursion formula for $G_k(\nu, \lambda)$ and other properties of these coefficients are given in [1]. If we write

$$
H_r^{(1)}(z) = J_r(z) + i Y_r(z)
$$

$$
= (2/\pi z)^{1/2} e^{i(z-\nu/2-\nu/4)} \sum_{k=0}^{\infty} H_k(\nu, \lambda) T_k(\nu/z), \quad z \geq \lambda,
$$

then the recurrence formula and other properties of the coefficients $H_k(\nu, \lambda)$ follow from those for $G_k(\nu, \lambda)$ upon replacing $\lambda$ by $\lambda e^{-i\pi/2}$. We have the normalization relation

$$
\sum_{r=0}^{\infty} (-1)^r H_r(\nu, \lambda) = 1
$$

and from [1], use of the backward recurrence relation for $H_r(\nu, \lambda)$ is convergent provided $|\operatorname{arg} \lambda| < \pi/2$.

3. Numerical Results. From (9) and (9) with a slight change of notation, we have

$$
J_r(z) = z^r \sum_{k=0}^{\infty} A_k(\nu) T_{2k}(\nu/8), \quad 0 < z \leq 8,
$$
Equation (15): \[ A_k(v) = \sum_{r=0}^{\infty} D_{r,k} T_r^*(\frac{v-s}{t}), \quad s \leq v \leq s + t. \]

In Tables 1 and 2, in the microfiche section of this issue, we present values of \( D_{r,k} \) which were computed by the technique depicted in [1] for \( s = 0, t = 4 \) and \( s = t = 4 \), respectively. Values of \( \Gamma(v + 1) \) required in the numerics were obtained by use of the schema of my previous paper [4]. Numerous checks were made on the coefficients. They are of the type previously discussed in [1], [2] and we dispense with further details. The computations were designed so that the coefficients for \( 0 \leq v \leq 4 \) are accurate to about 25D while those for \( 4 \leq v \leq 8 \) are accurate to about 27D. To evaluate \( J_\nu(z) \), we must incorporate the value of \( z' \). As \( 0 \leq z \leq 8 \), we see that the coefficients are sufficiently accurate to produce \( J_\nu(z) \) to about 20 decimals at least.

Now,

Equation (16): \[ Y_\nu(z) = (\csc \nu\pi)[(\cos \nu\pi) J_\nu(z) - J_{-\nu}(z)] \]

and both \( J_\nu(z) \) and \( Y_\nu(z) \) satisfy the same recurrence formula

Equation (17): \[ J_{\nu+1}(z) + J_{\nu-1}(z) = (2\nu/z)J_\nu(z). \]

Further, the recurrence formula for \( J_\nu(z) \) is always stable in the backward direction, but only conditionally stable in the forward direction. On the other hand, the recurrence formula for \( Y_\nu(z) \) is always stable in the forward direction. Thus, with the aid of the coefficients just described and the recurrence formulas, we can evaluate \( Y_\nu(z) \) for all \( z \) such that \( 0 \leq z \leq 8 \) and all \( \nu > 0 \), \( \nu \) an integer excepted. We have already referred to the availability of coefficients to compute \( Y_0(z) \) and \( Y_1(z) \). These together with the recurrence formula for \( Y_\nu(z) \) can be used to generate values of \( Y_n(z), n = 2, 3, \ldots \). As use of the recurrence formula in the forward direction for \( J_\nu(z) \) is limited, we leave for a future paper the development of Chebyshev coefficients for \( 0 \leq z \leq 8 \) and \( \nu > 8 \).

Using (12) with a slight change of notation, we write

Equation (18): \[ J_\nu(z) + i Y_\nu(z) = (2/\pi z)^{1/2} e^{i(\nu-\nu/2-\nu/4)} \sum_{k=0}^{\infty} H_k(\nu) T_k^*(5/z), \quad z \geq 5. \]

Let

Equation (19): \[ H_k(\nu) = \sum_{r=0}^{\infty} E_{r,k} T_r^*(\nu), \quad 0 \leq \nu \leq 1. \]

Table 3, also in the microfiche section of this issue, lists values of the real and imaginary parts of \( E_{r,k} \). These were obtained and checked by the methods previously described and we omit further details. The coefficients are sufficiently accurate to produce values of \( J_\nu(z) \) and \( Y_\nu(z) \) for \( \nu \) and \( z \) as noted to about 25 decimals. Since

Equation (20): \[ Y_{-\nu}(z) = (\cos \nu\pi) Y_\nu(z) + (\sin \nu\pi) J_\nu(z), \]

Equation (21): \[ J_{-\nu}(z) = (\cos \nu\pi) J_\nu(z) - (\sin \nu\pi) Y_\nu(z), \]

the coefficients \( E_{r,k} \) together with the recurrence formula for \( Y_\nu(z) \) enable the evaluation of \( Y_\nu(z) \) for all \( \nu > 0 \) and \( z \geq 5 \). A like statement cannot be made for \( J_\nu(z) \) as use of the recurrence formula in the forward direction for \( J_\nu(z) \) is limited. We defer
the development of coefficients to facilitate the evaluation of $J_v(z)$ when both $v$ and $z$ are large to a later paper.

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Mathematics Department
University of Missouri
Kansas City, Missouri 64110