Miniaturized Tables of Bessel Functions. III*

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Abstract. After the manner of our previous studies, coefficients for the expansion of
\( J_v(z) \) and \( Y_v(z) \) in double series of Chebyshev polynomials are presented. For \( J_v(z) \), the
ranges are (1) \( 0 < z \leq 8, 0 \leq v \leq 4 \), (2) \( 0 < z \leq 8, 4 \leq v \leq 8 \). For \( J_v(z) + iY_v(z) \), the
ranges are \( z \geq 5 \) and \( 0 \leq v \leq 1 \). The coefficients are given with sufficient accuracy to enable
the evaluation of the Bessel functions to at least 20 decimals.

1. Introduction. In previous studies \([1, 2]\), we considered the expansion of two
parameter functions in a double series of Chebyshev polynomials and developed
coefficients for the evaluation of \( K_v(z) \) and \( I_v(z) \) over a large part of the real \( z \) and
\( v \) lines. In the present paper, we give similar type coefficients for the evaluation of
\( J_v(z) \) and \( Y_v(z) \).

2. Chebyshev Expansion for \( J_v(z) \). From \([3, \text{Vol. 1, p. 212 and Vol. 2, p. 35}]\),

\[
J_v(z) = z^\nu \sum_{k=0}^\infty A_k(v, \lambda)T_{2k}(z/\lambda), \quad 0 < z \leq \lambda,
\]

(2)

\[
A_k(v, \lambda) = \frac{G_k(v, \lambda)}{2^k \Gamma(v + 1)},
\]

(3)

\[
G_k(v, \lambda) = \frac{\epsilon_k(-)^k \lambda^{2k} \Gamma(v + 1)}{2^{2k} k! \Gamma(k + v + 1)} \cdot \frac{\binom{\frac{1}{2} + k}{2k}}{\lambda^2/4},
\]

(4)

\[
2G_k(v, \lambda) = \frac{(k + 1)}{(k + 2)} \left\{ G_{k+1}(v, \lambda) - G_{k+2}(v, \lambda) \right\} - \frac{16(k + 1)(k + v + 1)}{\lambda^2} G_{k+1}(v, \lambda)
\]

\[
+ \left\{ 1 - \frac{16(k + 1)(k + 2 - v)}{\lambda^2} \right\} G_{k+2}(v, \lambda),
\]

where

(5) \( \epsilon_0 = 1, \epsilon_k = 2 \) for \( k > 0 \).

It is readily shown that

(6) \( G_k(v, \lambda) = \frac{\epsilon_k(-)^k \lambda^{2k} \Gamma(-\nu)}{2^{2k} (k!)^2} [1 + O(k^{-1})] \),

and for \( \nu \) and \( \lambda \) fixed,

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Thus, the expansion (1) converges and by letting \( z \to 0 \), we have the useful normalization relation

\[
\lim_{k \to \infty} G_k(\nu, \lambda) = 0.
\]

Further, after the manner of the discussion presented in [3, Vol. 2, pp. 159-166], we can show that use of the recursion formula (4) in the backward direction is convergent. Thus, for a fixed \( \lambda \), we can generate the coefficients \( A_k(\nu, \lambda) \) for any given value of \( \nu \). Then, following the discussion given in [1], we can develop coefficients \( D_{s,k}(\lambda) \) such that

\[
A_k(\nu, \lambda) = \sum_{s=0}^{\infty} D_{r,s}(\lambda) T_s^*(\frac{\nu - \frac{s}{t}}, \quad s \leq \nu \leq s + t.
\]

We remark that 20 decimal values of \( A_k(\nu, \lambda) \) are given in [3, pp. 331, 332, 352-356] for \( \lambda = 8 \) and \( \nu = 0, \pm \frac{1}{4}, \pm \frac{1}{2}, \pm \frac{3}{4}, \pm \frac{1}{3}, 1 \). Coefficients for the evaluation of \( Y_0(z) \) and \( Y_1(z) \) for \( 0 < z \leq 8 \) are also given in [3, pp. 331, 332].

We next present a descending type expansion in series of Chebyshev polynomials for the evaluation of \( J_\nu(z) \) and \( Y_\nu(z) \) in the vicinity of \( z = +\infty \). Now,

\[
H^{(1)}_\nu(z) = -\frac{2i}{\pi} e^{-i\pi/2} K_\nu(ze^{-i\pi/2}),
\]

and from [1], we have

\[
K_\nu(z) = (\pi/2z)^{1/2} e^{-z} \sum_{k=0}^{\infty} G_k(\nu, \lambda) T_k^*(\lambda/z), \quad \lambda \text{ fixed}, \lambda/z \leq 1, |\arg z| < \pi/2.
\]

The recursion formula for \( G_k(\nu, \lambda) \) and other properties of these coefficients are given in [1]. If we write

\[
H^{(1)}_\nu(z) = J_\nu(z) + i Y_\nu(z)
\]

then the recurrence formula and other properties of the coefficients \( H_\nu(\nu, \lambda) \) follow from those for \( G_k(\nu, \lambda) \) upon replacing \( \lambda \) by \( \lambda e^{-i\pi/2} \). We have the normalization relation

\[
\sum_{k=0}^{\infty} (-)^k H_k(\nu, \lambda) = 1
\]

and from [1], use of the backward recurrence relation for \( H_\nu(\nu, \lambda) \) is convergent provided \( |\arg \lambda| < \pi/2 \).

3. Numerical Results. From (1) and (9) with a slight change of notation, we have

\[
J_\nu(z) = z^\nu \sum_{k=0}^{\infty} A_k(\nu) T_{2k}(z/8), \quad 0 < z \leq 8,
\]
In Tables 1 and 2, in the microfiche section of this issue, we present values of $D_{r,k}$ which were computed by the technique depicted in [1] for $s = 0$, $t = 4$ and $s = t = 4$, respectively. Values of $\Gamma(\nu + 1)$ required in the numerics were obtained by use of the schema of my previous paper [4]. Numerous checks were made on the coefficients. They are of the type previously discussed in [1], [2] and we dispense with further details. The computations were designed so that the coefficients for $0 \leq \nu \leq 4$ are accurate to about 25D while those for $4 \leq \nu \leq 8$ are accurate to about 27D. To evaluate $J_\nu(z)$, we must incorporate the value of $z'$. As $0 \leq z \leq 8$, we see that the coefficients are sufficiently accurate to produce $J_\nu(z)$ to about 20 decimals at least.

Now,

\begin{equation}
Y_\nu(z) = (\csc \nu \pi)[(\cos \nu \pi) J_\nu(z) - J_{-\nu}(z)]
\end{equation}

and both $J_\nu(z)$ and $Y_\nu(z)$ satisfy the same recurrence formula

\begin{equation}
J_{\nu+1}(z) + J_{-\nu-1}(z) = (2\nu/z)J_\nu(z).
\end{equation}

Further, the recurrence formula for $J_\nu(z)$ is always stable in the backward direction, but only conditionally stable in the forward direction. On the other hand, the recurrence formula for $Y_\nu(z)$ is always stable in the forward direction. Thus, with the aid of the coefficients just described and the recurrence formulas, we can evaluate $Y_\nu(z)$ for all $z$ such that $0 \leq z \leq 8$ and all $\nu > 0$, $\nu$ an integer excepted. We have already referred to the availability of coefficients to compute $Y_\nu(z)$ and $Y_\nu(z)$. These together with the recurrence formula for $Y_\nu(z)$ can be used to generate values of $Y_n(z)$, $n = 2, 3, \ldots$. As use of the recurrence formula in the forward direction for $J_\nu(z)$ is limited, we leave for a future paper the development of Chebyshev coefficients for $0 \leq z \leq 8$ and $\nu > 8$.

Using (12) with a slight change of notation, we write

\begin{equation}
J_\nu(z) + iY_\nu(z) = (2/\pi z)^{1/2}e^{i(\nu - \nu/2 - \nu/4)} \sum_{k=0}^{\infty} H_k(\nu)T_k^*(5/z), \quad z \geq 5.
\end{equation}

Let

\begin{equation}
H_k(\nu) = \sum_{r=0}^{\infty} E_{r,k}T_r^*(\nu), \quad 0 \leq \nu \leq 1.
\end{equation}

Table 3, also in the microfiche section of this issue, lists values of the real and imaginary parts of $E_{r,k}$. These were obtained and checked by the methods previously described and we omit further details. The coefficients are sufficiently accurate to produce values of $J_\nu(z)$ and $Y_\nu(z)$ for $\nu$ and $z$ as noted to about 25 decimals. Since

\begin{equation}
Y_\nu(z) = (\cos \nu \pi) Y_\nu(z) + (\sin \nu \pi) J_\nu(z),
\end{equation}

\begin{equation}
J_{-\nu}(z) = (\cos \nu \pi) J_\nu(z) - (\sin \nu \pi) Y_\nu(z),
\end{equation}

the coefficients $E_{r,k}$ together with the recurrence formula for $Y_\nu(z)$ enable the evaluation of $Y_\nu(z)$ for all $\nu > 0$ and $z \geq 5$. A like statement cannot be made for $J_\nu(z)$ as use of the recurrence formula in the forward direction for $J_\nu(z)$ is limited. We defer
the development of coefficients to facilitate the evaluation of \( J_\nu(z) \) when both \( \nu \) and \( z \) are large to a later paper.

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