A Note Concerning the Two-Step Lax-Wendroff Method in Three Dimensions

By B. Eilon

Abstract. The two-step Lax-Wendroff method in three spatial dimensions is discussed and, dealing with its linear stability in the hydrodynamic case, the sufficiency of the von Neumann condition is proved.

In their paper [1], Rubin and Preiser suggest a difference scheme for the conservation law:

\[ W_t + \frac{\partial f_i}{\partial x_i} = 0. \]

Their scheme is an extension of Richtmyer's two-step method to three spatial dimensions. In order to deal with the linear stability they take the linearized equation:

\[ W_t + A_i \cdot \frac{\partial W}{\partial x_i} \] (where \( A_i = \frac{\partial f_i}{\partial W} \) are taken locally constant), and, in order to get a stability criterion, they compute the amplification matrix:

\[ G = I - \frac{3}{2} \lambda [ \cos \xi_1 + \cos \xi_2 + \cos \xi_3 ] M - 2 \lambda^2 M^2. \]

Here \( \lambda = \frac{\Delta t}{\Delta x_1} = \frac{\Delta t}{\Delta x_2} = \frac{\Delta t}{\Delta x_3}, \xi_i = k_i \Delta x_i \) (where \( k_i \) are the dual variables) and \( M = A_1 \sin \xi_1 + A_2 \sin \xi_2 + A_3 \sin \xi_3 \).

To prove sufficiency of the von Neumann condition, Rubin and Preiser use a theorem due to Kreiss [3] where the dissipativity of the scheme is assumed. However, it is easy to verify that their scheme is not dissipative because for \( \xi = (x, 0, 0) \) (so that \( |\xi| = \pi \)), for example, \( M \) is the null matrix and \( G = I \) so that its eigenvalues are on the unit circle, as is the case for two dimensions (see [4]).

We give a different proof for the sufficiency of the von Neumann condition but only for the hydrodynamic case. (This is an extension of Richtmyer's proof in [2] to three spatial dimensions.)

In this case, \( W = (\rho, \rho u, \rho v, \rho w, E) \), where \( \rho, E \) and \( V = (u, v, w) \) are the density, total energy per unit volume and the velocity vector, respectively. We shall make use of the following sufficiency theorem (see [2]): "If \( G \) has a complete set of eigenvectors and there exists a constant \( \delta \) such that \( \Delta \geq \delta > 0 \), where \( \Delta^2 \) is the Gram determinant of the normalized eigenvectors, then the von Neumann condition is sufficient as well as necessary for stability".

Instead of calculating the eigenvectors of \( G \), we shall consider another matrix \( G' \) obtained from \( G \) by a similarity transformation. We introduce \( W' = (\rho, u, v, w, p) \), where \( p \) is the pressure, and the transformation is...
b. \( e \) is too complicated.

If we compute \( P \) from (2.a), we get that

\[
det P = \rho \frac{\partial e}{\partial p},
\]

where \( e \) is the internal energy per unit mass. It turns out that \( P \) is triangular so that \( \det(P^{-1}) = \det(P)^{-1} \).

Let \( y_i \) be the normalized eigenvectors of \( G \) (and \( M \)), then \( \alpha_i P^{-1} y_i \) are the normalized eigenvectors of \( G' \) (and \( M' \)), where \( \alpha_i > 0 \) are the normalizing factors.

If we define

\[
\Delta_1 = |\det(y_1, y_2, \cdots, y_n)|,
\]

\[
\Delta_2 = |\det(\alpha_1 P^{-1} y_1, \alpha_2 P^{-1} y_2, \cdots, \alpha_n P^{-1} y_n)|,
\]

and \( \alpha = \alpha_1 \alpha_2 \cdots \alpha_n > 0 \), then

\[
\Delta_2 = |\det(\alpha P^{-1} y_1, y_2, \cdots, y_n)|
= \alpha |\det P^{-1} \cdot |\det(y_1, y_2, \cdots, y_n)|
= \alpha |\det P^{-1} | \Delta_1 = \frac{\alpha}{\rho \frac{\partial e}{\partial p}} \Delta_1.
\]

This is the case because \( \rho \) is always bounded away from zero and in the usual fluids the same holds for \( \frac{\partial e}{\partial p} \) and consequently for \( \det P^{-1} \). Therefore, \( \Delta_1 \) is bounded away from zero if and only if \( \Delta_2 \) is.

Rubin and Preiser found \( M' \) to be

\[
M' = \begin{bmatrix}
u' & \rho \cos r & \rho \cos s & \rho \cos t & 0 \\
0 & u' & 0 & 0 & 1/\rho \cos r \\
0 & 0 & u' & 0 & 1/\rho \cos s \\
0 & 0 & 0 & u' & 1/\rho \cos t \\
0 & \rho \cos^2 \cos r & \rho \cos^2 \cos s & \rho \cos^2 \cos t & u'
\end{bmatrix}
\]

where \( L = (\sin^2 \xi_1 + \sin^2 \xi_2 + \sin^2 \xi_3)^{1/2} \), \( \cos r = (\sin \xi_1)/L \), \( \cos s = (\sin \xi_2)/L \), \( \cos t = (\sin \xi_3)/L \) and \( u' = u \cos r + v \cos s + w \cos t \).

A direct computation of its eigenvectors shows

\[
\Delta_2 = \det
\begin{bmatrix}
1 & 0 & 0 & -k_p/\cos r & k_p/\cos r \\
0 & -\cos r \cdot \cot s & -\cos t / \sin s & k & k \\
0 & \sin s & 0 & k \cos s / \cos r & k \cos s / \cos r \\
0 & -\cos t \cdot \cot s & \cos r / \sin s & k \cos t / \cos r & k \cos t / \cos r \\
0 & 0 & 0 & -k_p/cos r & k_p/\cos r
\end{bmatrix}
\]

\[
= \frac{2 \rho C k^2}{\cos^3 r}.
\]
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\[ k^2 = \frac{(\cos^2 r)/(\rho^2 C^2 + 1 + \rho^2/C^2)}{\text{is a normalizing factor so that finally}} \]

\[ \Delta_2 = \frac{2\rho C}{(\rho^2 C^2 + 1 + \rho^2/C^2)}. \]

We see that \( \Delta_2 \), and so \( \Delta_1 \), is bounded away from zero. Hence, by the sufficiency theorem quoted above, the von Neumann condition, namely \( \Delta t/\Delta x \leq 1/\sqrt{3(|V| + C)} \), implies linear stability.

Department of Mathematical Sciences
Tel-Aviv University
Tel-Aviv, Israel