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On Evaluation of Moments of $K_\nu(t)/I_\nu(t)$

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Abstract. This paper presents a method of evaluation of the moments of $K_\nu(t)/I_\nu(t)$. Two pairs of expressions, each consisting of two series, are obtained according to the index being an even or an odd integer. The method is an extension of the method used by Watson. Values are tabulated to 12D for $\nu = 0(1)2$.

In a recent paper [1], Roberts considered the computation of the integral

$$M_k^{(\nu)} = \int_0^\infty t^k \frac{K_\nu(t)}{I_\nu(t)}\, dt \qquad (k \geqq 2\nu), \tag{1}$$

where I_ν and K_ν are modified Bessel functions. He developed the integral into an 'asymptotic series' suitable for computation when k is a large integer. When k is otherwise a small integer, he integrated numerically the following equivalent integral by using Simpson's rule:

$$M_k^{(\nu)} = \frac{1}{k+1} \int_0^\infty \frac{t^k}{I_\nu^2(t)}\, dt \qquad (k \geqq 2\nu). \tag{2}$$

In particular, values are tabulated to 8S for $\nu = 1$ and $k = 2(1)100$.

Some time ago, Watson [2] evaluated the integral in (2) when k is an even integer. He developed the integral into two series by employing a method based on a modification of Plana's summation formula. It is found that this method can be extended to the case when k is an odd integer. Furthermore, it is also found that the same integral can be developed into different series by a modification of the method. Altogether, two pairs of expressions are obtained according to k being an even or an odd integer. It is the purpose of this paper to present such results. Values of the integral are thereby evaluated to 12D for $\nu = 0(1)2$. It is mentioned that a method analogous to the present one was used recently by the authors for the evaluation of two Howland integrals [3].

For convenience, the integral is redefined, together with a factor, as follows:

$$L_k^{(\nu)} = \frac{2^{k+1}}{\pi(k!)} \int_0^\infty t^k \frac{K_\nu(t)}{I_\nu(t)}\, dt \qquad (k \geqq 2\nu) \tag{3}$$

so that it tends asymptotically to unity as k tends to infinity. The equivalent integral is

$$L_k^{(\nu)} = \frac{2^{k+1}}{\pi(k+1)!} \int_0^\infty \frac{t^k}{I_\nu^2(t)}\, dt \qquad (k \geqq 2\nu). \tag{4}$$

The following results are obtained for $k \geqq \nu$:

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$$L_{2k}^{(\nu)} = \frac{2^{2k+1}}{(2k+1)!}\left[\frac{2^{2\nu-1}(\nu!)^2\alpha}{\pi}\delta_{\nu,k} + \frac{\alpha}{\pi}\sum_{n=1}^{\infty}\frac{(n\alpha)^{2k}}{I_\nu^2(n\alpha)} + (-1)^{\nu+k}\sum_{m=1}^{\infty}\frac{j_m^{2k-1}\exp(-a_m)}{J_{\nu+1}^2(j_m)\sinh a_m}\cdot\{2k+1-a_m(1+\coth a_m)\}\right],$$

(5)

$$L_{2k+1}^{(\nu)} = \frac{2^{2k+2}}{\pi(2k+2)!}\left[\frac{2\alpha}{\pi}\sum_{n=1}^{\infty}\frac{(n\alpha)^{2k+1}}{I_\nu^2(n\alpha)}\operatorname{Si}(n\pi) - (-1)^{\nu+k}\sum_{m=1}^{\infty}\frac{j_m^{2k}}{J_{\nu+1}^2(j_m)\sinh a_m}\cdot\{(2k+2-a_m\coth a_m)E_3(a_m)+a_mE_2(a_m)\}\right],$$

and

$$L_{2k}^{(\nu)} = \frac{2^{k+1}}{(2k+1)!}\left[\frac{\alpha}{\pi}\sum_{n=0}^{\infty}\frac{(n\alpha+\frac{1}{2}\alpha)^{2k}}{I_\nu^2(n\alpha+\frac{1}{2}\alpha)} - (-1)^{\nu+k}\sum_{m=1}^{\infty}\frac{j_m^{2k+1}\exp(-a_m)}{J_{\nu+1}^2(j_m)\cosh a_m}\cdot\{2k+1-a_m(1+\tanh a_m)\}\right],$$

(6)

$$L_{2k+1}^{(\nu)} = \frac{2^{2k+2}}{\pi(2k+2)!}\left[\frac{2\alpha}{\pi}\sum_{n=0}^{\infty}\frac{(n\alpha+\frac{1}{2}\alpha)^{2k+1}}{I_\nu^2(n\alpha+\frac{1}{2}\alpha)}\operatorname{Si}(n\pi+\tfrac{1}{2}\pi) + (-1)^{\nu+k}\sum_{m=1}^{\infty}\frac{j_m^{2k}}{J_{\nu+1}^2(j_m)\cosh a_m}\cdot\{2-(2k+2-a_m\tanh a_m)E_2(a_m)-a_mE_3(a_m)\}\right].$$

The derivation will be described in the Appendix. The first expression in (6) is the one obtained before by Watson while the other three are all new. In these expressions, α is a positive constant, $\delta_{\nu,k}$ is Kronecker delta, j_m is mth positive zero of the Bessel function $J_\nu(z)$, and a_m is

(7) $$a_m = \pi j_m/\alpha.$$

In addition, Si is the sine integral, and E_2 and E_3 are

(8) $$\begin{matrix}E_2\\E_3\end{matrix}(a) = E_1(a)e^{a} \pm E_1(-a)e^{-a},$$

where E_1 is exponential integral defined by

(9) $$E_1(a) = \int_a^\infty \frac{e^{-t}}{t}\,dt.$$

The integral can therefore be computed from one pair of the expressions and checked by the other pair. It is seen that each expression consists of two series. The constant α is involved on the right of each expression only. This constant can be fixed to suit our convenience. The first series converges rapidly when α is large and the second series when α is small. The first series of the first expression in (5) represents, in fact, the value of the integral given by trapezoidal rule while the first series of the first expression in (6) represents the value given by rectangular rule. In each such series, α stands for the increment. Thus, the second series may be regarded merely as a correction. Its value can be made small by a proper choice of α. In the present computation, α will be chosen as 2/5.

The values of $\mathrm{Si}(n\pi/2)$ have been tabulated recently, together with a factor $2/\pi$, by the authors [4] to 25D for $n = 1(1)200$. Further values can be generated readily whenever needed. The computation of $I_\nu(n\pi/2)$ from its series expansion is straightforward. For $\nu = 2$ and $k = 50$, to attain an accuracy of 12D, 160 terms of the first series are needed when $\alpha = 2/5$ while 65 terms are needed when $\alpha = 1$. To attain an accuracy of 8D, the corresponding terms needed are 137 and 56, respectively. The convergence is more rapid when k is smaller. On the other hand, when ν is smaller, it seems that there is no appreciable effect on the convergence.

The readily accessible values of j_m and $J_{\nu+1}(j_m)$ are the 10D tables for $\nu = 0$ and 1, [5], [6], and the 8D tables for $\nu = 0(1)20$, [7]. The series expansion of E_1 is

$$E_1(\pm a) = -\gamma - \log a - \sum_{m=1}^{\infty} \frac{(\mp a)^m}{m(m!)} \qquad (a > 0), \tag{10}$$

where γ is Euler's constant. Or, when a is large, it is given by the asymptotic series:

$$\begin{aligned} E_1(a)e^a &\sim \frac{1}{a}\left(1 - \frac{1}{a} + \frac{2!}{a^2} - \frac{3!}{a^3} + \cdots\right), \\ E_1(-a)e^{-a} &\sim -\frac{1}{a}\left(1 + \frac{1}{a} + \frac{2!}{a^2} + \frac{3!}{a^3} + \cdots\right), \end{aligned} \tag{11}$$

from which

$$\begin{aligned} E_2(a) &\sim -\frac{2}{a^2}\left(1 + \frac{3!}{a^2} + \frac{5!}{a^4} + \cdots\right), \\ E_3(a) &\sim \frac{2}{a}\left(1 + \frac{2!}{a^2} + \frac{4!}{a^4} + \cdots\right). \end{aligned} \tag{12}$$

To attain a resulting accuracy of 12D, the existing values of j_m are adequate to compute $E_2(a_m)$ and $E_3(a_m)$ from the asymptotic series in (12) for $m \geqq 2$ when $\alpha = 2/5$. For $m = 1$, however, a more accurate value is needed because E_2 and E_3 are now to be computed through E_1 from the series expansion in (10). The accuracy of this value of j_m can be improved readily by using Newton-Raphson method. Generally, two or three terms only are needed to compute the second series when $\alpha = 2/5$. If $\alpha = 1$, however, more terms are needed and the existing values of j_m are adequate only for $m \geqq 3$. The Newton-Raphson method used to improve the accuracy of j_m becomes less convenient as m increases.

The computation is carried out on an IBM 360 computer with $\alpha = 2/5$, except that j_1 and $E_1(a_1)$ are computed on an IBM 1620 computer. The values obtained from (5) and (6) are in full agreement as anticipated. The results rounded to 12D for $\nu = 0(1)2$ and $k = 0(1)50$ are shown in Table 1. Further values, whenever needed, can be

TABLE 1. Values of $L_k^{(\nu)}$.

k	$\nu = 0$	$\nu = 1$	$\nu = 2$
0	0.87069 01325 38	–	–
1	0.80390 09176 36	–	–
2	0.82364 49850 49	3.18729 66891 95	–
3	0.85267 52682 33	1.97801 02455 82	–
4	0.87847 57703 09	1.60062 71295 52	12.31547 04126 53
5	0.89906 50568 53	1.42465 66174 98	6.09445 76146 44
6	0.91494 61624 33	1.32591 89429 31	4.17452 23208 14
7	0.92711 65811 13	1.26392 86310 12	3.27283 32328 61
8	0.93650 84785 70	1.22184 78667 45	2.75859 45107 88
9	0.94385 80976 67	1.19156 64696 98	2.42966 73332 30
10	0.94970 94221 04	1.16876 78085 94	2.20256 45154 99
11	0.95445 29526 51	1.15097 78232 74	2.03694 48709 32
12	0.95836 63078 82	1.13669 29783 64	1.91109 05127 11
13	0.96164 71845 27	1.12495 40894 53	1.81234 15511 72
14	0.96443 76170 71	1.11512 33486 89	1.73285 18324 33
15	0.96684 10189 24	1.10676 11271 21	1.66751 51916 36
16	0.96893 38874 45	1.09955 47411 12	1.61287 31945 11
17	0.97077 37756 96	1.09327 56161 41	1.56650 40403 29
18	0.97240 47241 59	1.08775 26147 17	1.52666 32415 86
19	0.97386 09860 22	1.08285 48986 86	1.49206 33400 85
20	0.97516 96095 52	1.07848 05980 34	1.46173 39611 02
21	0.97635 22522 70	1.07454 91205 10	1.43493 01218 73
22	0.97742 64745 97	1.07099 58064 45	1.41107 05126 70
23	0.97840 66767 63	1.06776 81323 06	1.38969 49505 37
24	0.97930 47879 89	1.06482 29608 69	1.37043 44206 02
25	0.98013 07810 87	1.06212 45139 11	1.35298 95757 18
26	0.98089 30622 16	1.05964 28535 02	1.33711 50395 93
27	0.98159 87699 92	1.05735 27278 07	1.32260 77676 35
28	0.98225 40078 71	1.05523 26824 14	1.30929 82939 23
29	0.98286 40267 00	1.05326 43679 88	1.29704 40633 04
30	0.98343 33696 07	1.05143 19950 68	1.28572 42916 39
31	0.98396 59880 76	1.04972 19005 24	1.27523 59608 14
32	0.98446 53357 39	1.04812 21997 05	1.26549 06666 68
33	0.98493 44447 61	1.04662 25050 33	1.25641 21151 93
34	0.98537 59885 03	1.04521 36966 24	1.24793 41166 19
35	0.98579 23332 67	1.04388 77339 88	1.23999 89655 60
36	0.98618 55813 04	1.04263 75004 40	1.23255 61232 14
37	0.98655 76067 61	1.04145 66737 64	1.22556 11378 69
38	0.98691 00858 85	1.04033 96180 70	1.21897 47548 99
39	0.98724 45225 42	1.03928 12929 01	1.21276 21785 34
40	0.98756 22698 74	1.03827 71764 44	1.20689 24560 32
41	0.98786 45487 60	1.03732 32003 54	1.20133 79612 06
42	0.98815 24636 24	1.03641 56941 91	1.19607 39590 89
43	0.98842 70160 24	1.03555 13378 43	1.19107 82372 46

TABLE 1. Values of $L_k^{(\nu)}$ (continued).

k	$\nu = 0$	$\nu = 1$	$\nu = 2$
44	0.98868 91163 76	1.03472 71206 31	1.18633 07921 13
45	0.98893 95941 07	1.03394 03060 21	1.18181 35610 11
46	0.98917 92064 79	1.03318 84010 68	1.17751 01922 75
47	0.98940 86462 86	1.03246 91298 61	1.17340 58472 89
48	0.98962 85485 74	1.03178 04103 77	1.16948 70294 05
49	0.98983 94965 48	1.03112 03342 34	1.16574 14355 57
50	0.99004 20267 61	1.03048 71489 39	1.16215 78271 49

TABLE 2. Values of $c_n^{(\nu)}$.

n	$\nu = 0$	$\nu = 1$	$\nu = 2$
1	−0.50000 00000 00	1.50000 00000 00	7.50000 00000 00
2	0.06250 00000 00	0.56250 00000 00	14.06250 00000 00
3	−0.17708 33333 33	0.53125 00000 00	12.65625 00000 00
4	0.02180 98958 33	0.17285 15625 00	7.25097 65625 00
5	−0.11285 80729 17	0.22294 92187 50	2.19287 10937 50
6	0.01009 79275 2	0.05631 10351 6	−0.05218 50585 9
7	−0.08359 03592	0.12678 48424	−0.71666 17257
8	0.00555 5340	0.02454 2529	−0.52561 0277
9	−0.06640 809	0.08858 257	−0.40423 078
10	0.00346 03	0.01359 42	−0.20316 24
11	−0.05504 6	0.06860 2	−0.18946 0
12	0.00236	0.00870	−0.09267
13	−0.0470	0.0562	−0.1187

computed either from the asymptotic series for the original integral given by Roberts or from a similar series for the equivalent integral given by Brenner and Sonshine [8]. By referring to the former, the series after a slight modification becomes

$$L_k^{(\nu)} \sim 1 + c_1^{(\nu)} \Big/ \binom{k}{1} + c_2^{(\nu)} \Big/ \binom{k}{2} + c_3^{(\nu)} \Big/ \binom{k}{3} + \cdots . \tag{13}$$

Note that here $c_n^{(\nu)}$ is Roberts' b_n divided by $n!$. The first 13 coefficients of $c_n^{(\nu)}$ for $\nu = 0(1)2$ are given in Table 2. They are adequate to generate values of the integral to 12D when $k \geqq 50$.

As mentioned before, when $\nu = 1$, the values of $M_k^{(1)}$ were tabulated by Roberts to 8S for $k = 2(1)100$. When the present values are converted and compared, it is found that Roberts' values are generally correct, save for a frequent round-off error of one unit in the last digit. When $\nu = 0$, the values of $(2k + 1)M_{2k}^{(0)}/(k!)^2$ were tabulated recently by Smythe [9] to 8S for $k = 0(1)83$. When the present values are converted and compared, it is found that the Smythe's values generally err in the last digit or occasionally in the seventh digit by one unit. When $\nu = 2$, the values of $L_k^{(2)}$ were computed before by the first author [10] to 6S for $k = 4(2)24$. Comparison shows that the previous values generally err in the fifth or sixth digit, save the first two which err in the third digit.

It is noted that the values of the integral can be used to evaluate allied integrals. For example, consider the following integrals:

$$
\text{(14)}\qquad
\begin{aligned}
&\begin{matrix} S_k^{(\nu)} \\ C_k^{(\nu)} \end{matrix}(a) = \int_0^\infty t^k \frac{K_\nu(t)}{I_\nu(t)} \begin{matrix} \sin \\ \cos \end{matrix}\, at\, dt \qquad \begin{matrix} (k + 1 \geqq 2\nu), \\ (k \geqq 2\nu), \end{matrix} \\
&\begin{matrix} I_k^{(\nu,m,n)} \\ J_k^{(\nu,m,n)} \end{matrix}(a) = \int_0^\infty t^k \frac{K_\nu(t)}{I_\nu(t)} \begin{matrix} I_m \\ J_m \end{matrix}(at) \begin{matrix} I_n \\ J_n \end{matrix}(at)\, dt \qquad (k + m + n \geqq 2\nu),
\end{aligned}
$$

where a is restricted to be positive and less than unity in the third integral but not restricted in the other integrals. Note that the third integral is in fact a more general integral than the one considered by Sneddon [11, p. 138]. With the aid of the following expansions [12, p. 147],

$$
\text{(15)}\qquad \begin{matrix} I_m \\ J_m \end{matrix}(at) \begin{matrix} I_n \\ J_n \end{matrix}(at) = \sum_{p=0}^\infty \frac{(\pm 1)^p}{(m + p)!\,(n + p)!} \binom{m + n + 2p}{p} \left(\frac{at}{2}\right)^{m+n+2p},
$$

and also with those of sine and cosine, the preceding integrals are developed into the following series:

$$
\text{(16)}\qquad
\begin{aligned}
S_k^{(\nu)}(a) &= \frac{\pi(k!)}{2^{k+1}} \sum_{p=0}^\infty (-1)^p \binom{k + 2p + 1}{2p + 1} \left(\frac{a}{2}\right)^{2p+1} L_{k+2p+1}^{(\nu)}, \\
C_k^{(\nu)}(a) &= \frac{\pi(k!)}{2^{k+1}} \sum_{p=0}^\infty (-1)^p \binom{k + 2p}{2p} \left(\frac{a}{2}\right)^{2p} L_{k+2p}^{(\nu)}, \\
\begin{matrix} I_k^{(\nu,m,n)} \\ J_k^{(\nu,m,n)} \end{matrix}(a) &= \frac{\pi(k!)}{2^{k+1}} \sum_{p=0}^\infty (\pm 1)^p \binom{k + m + n + 2p}{k} \binom{m + n + 2p}{p} \\
&\qquad \cdot \binom{m + n + 2p}{m + p} \left(\frac{a}{4}\right)^{m+n+2p} L_{k+m+n+2p}^{(\nu)}.
\end{aligned}
$$

A few values thus computed are shown below:

ν	k	a	$S_k^{(\nu)}(a)$	$C_k^{(\nu)}(a)$
0	1	1	0.40184 17420	0.29000 20075
1	2	1	1.39176 0847	1.29252 2901
2	4	1	16.44405 746	9.45912 7980

ν	m	n	k	a	$I_k^{(\nu,m,n)}(a)$	$J_k^{(\nu,m,n)}(a)$
0	1	1	1	4/5	0.58980 63520	0.08296 29247
1	1	1	2	4/5	3.28690 1939	0.28680 7351
2	1	1	4	4/5	213.75061 70	4.19774 356

Appendix. To derive (5), consider the contour integral

$$
\text{(A.1)}\qquad \frac{1}{2\pi i} \oint \frac{z^k\, dz}{(z - t) I_\nu^2(z) \sin(\pi z/\alpha)} \qquad (k \geqq 2\nu),
$$

where the contour is taken around the circle $|z| = R$ through a sequence of values such that the circle never passes through any pole of the integrand. t is any point inside the circle. This integral tends to zero as R tends to infinity. The poles of the integrand are

$$z = t, \quad z = \pm n\alpha, \quad z = \pm i j_m, \tag{A.2}$$

where $n = 1, 2, 3, \cdots$ and $m = 1, 2, 3, \cdots$. In particular, when $k = 2\nu$, the origin $z = 0$ is also a pole.

It follows from Cauchy's theorem of residues that the sum of residues at all the poles is zero. Consequently, the residue at $z = t$ is

$$\begin{aligned}\frac{t^k}{I_\nu^2(t)\sin(\pi t/\alpha)} &= \frac{2^{2\nu}(\nu!)^2\alpha}{\pi t}\delta_{2\nu,k} - \frac{\alpha}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n(n\alpha)^k}{I_\nu^2(n\alpha)}\left\{\frac{1}{n\alpha - t} - \frac{(-1)^k}{n\alpha + t}\right\}\\ &\quad + (-1)^\nu\sum_{m=1}^{\infty}\frac{(ij_m)^{k-1}}{J_{\nu+1}^2(j_m)\sinh a_m}\\ &\qquad\cdot\left[j_m\left\{\frac{1}{(j_m + it)^2} - \frac{(-1)^k}{(j_m - it)^2}\right\}\right.\\ &\qquad\left. - (k + 1 - a_m\coth a_m)\left\{\frac{1}{j_m + it} - \frac{(-1)^k}{j_m - it}\right\}\right].\end{aligned} \tag{A.3}$$

Multiply both sides by $\sin(\pi t/\alpha)$ and integrate with respect to t from zero to infinity. When k is an even integer, the values of the three integrals on the right are

$$\begin{aligned}\int_0^\infty \frac{t\sin(\pi t/\alpha)\,dt}{n^2\alpha^2 - t^2} &= -(-1)^n\frac{\pi}{2},\\ \int_0^\infty \frac{t\sin(\pi t/\alpha)\,dt}{j_m^2 + t^2} &= \frac{\pi}{2}\exp(-a_m),\\ \int_0^\infty \frac{t\sin(\pi t/\alpha)\,dt}{(j_m^2 + t^2)} &= \frac{\pi^2}{4\alpha j_m}\exp(-a_m).\end{aligned} \tag{A.4}$$

When k is an odd integer, they are

$$\begin{aligned}\int_0^\infty \frac{\sin(\pi t/\alpha)\,dt}{n^2\alpha^2 - t^2} &= -\frac{(-1)^n}{n\alpha}\operatorname{Si}(n\pi),\\ \int_0^\infty \frac{\sin(\pi t/\alpha)\,dt}{j_m^2 + t^2} &= \frac{1}{2j_m}E_3(a_m),\\ \int_0^\infty \frac{j_m^2 - t^2}{(j_m^2 + t^2)^2}\sin\frac{\pi t}{\alpha}\,dt &= -\frac{\pi}{2\alpha}E_2(a_m).\end{aligned} \tag{A.5}$$

With these values, the two expressions in (5) are derived.

Next, to derive (6), consider the contour integral

$$\frac{1}{2\pi i}\oint\frac{z^k\,dz}{(z - t)I_\nu^2(z)\cos(\pi z/\alpha)} \qquad (k \geqq 2\nu), \tag{A.6}$$

where the same contour is taken as before. The poles of the integrand are

$$z = t, \quad z = \pm(n + \tfrac{1}{2})\alpha, \quad z = \pm i j_m, \tag{A.7}$$

where $n = 0, 1, 2, 3, \cdots$ and $m = 1, 2, 3, \cdots$. Likewise, from Cauchy's theorem of residues, the residue at $z = t$ is

$$\tag{A.8} \begin{aligned} \frac{t^k}{I_\nu^2(t)\cos(\pi t/\alpha)} &= \frac{\alpha}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n(n\alpha+\frac{1}{2}\alpha)^k}{I_\nu^2(n\alpha+\frac{1}{2}\alpha)}\left\{\frac{1}{n\alpha+\frac{1}{2}\alpha-t}+\frac{(-1)^k}{n\alpha+\frac{1}{2}\alpha+t}\right\} \\ &\quad + (-1)^\nu \sum_{m=1}^{\infty}\frac{i^k j_m^{k-1}}{J_{\nu+1}^2(j_m)\cosh a_m} \\ &\qquad \cdot\left[j_m\left\{\frac{1}{(j_m+it)^2}+\frac{(-1)^k}{(j_m-it)^2}\right\}\right. \\ &\qquad\quad \left. -(k+1-a_m\tanh a_m)\left\{\frac{1}{j_m+it}+\frac{(-1)^k}{j_m-it}\right\}\right]. \end{aligned}$$

Multiply both sides by $\cos(\pi t/\alpha)$ and integrate with respect to t from zero to infinity. According to k being an even or an odd integer, the values of the three integrals on the right are, respectively,

$$\tag{A.9} \begin{aligned} \int_0^\infty \frac{\cos(\pi t/\alpha)\,dt}{(n\alpha+\frac{1}{2}\alpha)^2-t^2} &= (-1)^n\frac{\pi}{(2n+1)\alpha}, \\ \int_0^\infty \frac{\cos(\pi t/\alpha)\,dt}{j_m^2+t^2} &= \frac{\pi}{2j_m}\exp(-a_m), \\ \int_0^\infty \frac{j_m^2-t^2}{(j_m^2+t^2)^2}\cos\frac{\pi t}{\alpha}\,dt &= \frac{\pi^2}{2\alpha}\exp(-a_m), \end{aligned}$$

and

$$\tag{A.10} \begin{aligned} \int_0^\infty \frac{t\cos(\pi t/\alpha)\,dt}{(n\alpha+\frac{1}{2}\alpha)^2-t^2} &= (-1)^n\,\mathrm{Si}(n\pi+\tfrac{1}{2}\pi), \\ \int_0^\infty \frac{t\cos(\pi t/\alpha)\,dt}{j_m^2+t^2} &= \tfrac{1}{2}E_2(a_m), \\ \int_0^\infty \frac{t\cos(\pi t/\alpha)\,dt}{(j_m^2+t^2)^2} &= \frac{1}{2j_m^2}-\frac{a_m}{4j_m^2}E_3(a_m). \end{aligned}$$

Thence, the two expressions in (6) are derived.

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