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One-Step Piecewise Polynomial Galerkin Methods for Initial Value Problems*

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Abstract. A new approach to the numerical solution of systems of first-order ordinary differential equations is given by finding local Galerkin approximations on each subinterval of a given mesh of size h. One step at a time, a piecewise polynomial, of degree n and class C^0, is constructed, which yields an approximation of order $O(h^{2n})$ at the mesh points and $O(h^{n+1})$ between mesh points. In addition, the jth derivatives of the approximation on each subinterval have errors of order $O(h^{n-j+1})$, $1 \leqq j \leqq n$. The methods are related to collocation schemes and to implicit Runge-Kutta schemes based on Gauss-Legendre quadrature, from which it follows that the Galerkin methods are A-stable.

1. Introduction. In this paper, we show how Galerkin's method can be employed to devise one-step methods for systems of nonlinear first-order ordinary differential equations. The basic idea is to find local nth degree polynomial Galerkin approximations on each subinterval of a given mesh and to match them together continuously, but not smoothly.

For each $n \geqq 1$, a method is defined (Section 2) which uses an n-point Gauss-Legendre quadrature formula to evaluate certain inner products in the Galerkin equations. For sufficiently small step size h, a unique numerical solution exists and may be found by successive substitution (Section 3). After showing that these Galerkin methods are also collocation methods (Section 4) and implicit Runge-Kutta methods (Section 5), we show that the mesh point errors are of the order $O(h^{2n})$, and the global errors are of the order $O(h^{n+1})$ for the approximate solution and $O(h^{n-j+1})$, $1 \leqq j \leqq n$, for its jth derivatives (Section 6). A proof of the A-stability of the methods is given in Section 7, and numerical results are presented in Section 8.

Discrete one-step methods based on quadrature, other than the classical Runge-Kutta methods, have been studied by several authors, including the explicit schemes in [12, p. 101], [13], [14], [22] and the implicit schemes in [1], [2], [3], [6, Chapters 4, 9], [10], [12, p. 159]. Also, discrete block implicit methods are given in [21], [24], [25]. The methods of this paper, however, yield continuous piecewise polynomial approximations with the inherent benefit of derivative approximations. Earlier uses of piecewise polynomials may be found in [4], [5], [11], [15], [16], [17], [26].

Finally, we remark that recent "semidiscrete" Galerkin methods [7], [9], [18], [19], [23] reduce initial-boundary value problems to systems of ordinary differential equations. When combined with such methods, our techniques open the possibility of "fully discrete" Galerkin methods for these problems.

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2. Piecewise Polynomial Galerkin Methods. We consider the numerical solution of only a single nonlinear ordinary differential equation

$$u'(t) = f(t, u(t)), \qquad t_0 \leqq t, \tag{1}$$

$$u(t_0) = u_0 \tag{2}$$

on a finite interval $[t_0, t_N]$, although the results carry over to systems of such equations. We assume that $f(t, x) \in C^{2n}$ in $[t_0, t_N] \times (-\infty, \infty)$, so that the exact solution $u(t) \in C^{2n+1}[t_0, t_N]$, $n \geqq 1$, and we also assume that f has a Lipschitz constant L in this same region.

Let π: $t_i = t_0 + ih$, $0 \leqq i \leqq N$, be a *uniform mesh* for the sake of simplicity. (It will be seen that our arguments do not depend crucially on this assumption since our method is a one-step method and step size changes are easy.) Then, we may approximate $u(t)$ on each subinterval by an *nth degree polynomial*

$$y(t) = \sum_{j=1}^{n+1} b_j^{(i)} \varphi_{i+j}(t), \qquad t_i \leqq t \leqq t_{i+1}, \; 0 \leqq i \leqq N-1, \tag{3}$$

where $\varphi_{i+j}(t)$ are basis functions which are nth degree polynomials on each $[t_i, t_{i+1}]$. For example, $\{\varphi_k\}_{k=1}^{N+n}$ might be the nth degree B-spline basis functions of Schoenberg [20] or some other piecewise polynomial basis. Since the b_j may change from one subinterval to the next, $y(t)$ need not be as smooth as the $\varphi_k(t)$.

We require that $y(t)$ be *continuous* on $[t_0, t_N]$ and that it provide a *local Galerkin approximation* on each subinterval $[t_i, t_{i+1}]$, $0 \leqq i \leqq N-1$. Accordingly, on each subinterval, we write the following $n+1$ equations (one linear, n nonlinear) for the $b_j^{(i)}$, $1 \leqq j \leqq n+1$,

$$\begin{aligned} y_{i+} &= y_{i-}, \qquad i \geqq 1, \\ &= u_0, \qquad i = 0, \end{aligned} \tag{4}$$

$$(y' - f(t, y), \varphi_{i+k})_i = 0, \qquad 2 \leqq k \leqq n+1, \; 0 \leqq i \leqq N-1, \tag{5}$$

using the notation

$$(v, w)_i = \int_{t_i}^{t_{i+1}} v(t) w(t) \, dt.$$

To obtain a computational form of (4)–(5), we assume that the $(\varphi'_{i+j}, \varphi_{i+k})_i$ in (5) are computed exactly, i.e., analytically or by an exact quadrature formula, while the inner products $(f, \varphi_{i+k})_i$ are replaced by the *n-point Gauss-Legendre quadrature* formula having the form

$$\int_{t_i}^{t_{i+1}} v(t) \, dt = h \sum_{k=1}^{n} w_k v(\sigma_{i,k}) + O(h^{2n+1}), \tag{6}$$

$$\sigma_{i,k} = t_i + \theta_k h, \qquad 1 \leqq k \leqq n, \tag{7}$$

where $w_k > 0$ and θ_k are the weights and abscissae for $[0, 1]$. The result is that (4)–(5) are replaced by the following set of N systems of $n+1$ nonlinear equations to be solved in succession

$$\mathbf{A}\mathbf{b}^{(i)} = \mathbf{c}^{(i)}(\mathbf{b}^{(i)}), \qquad 0 \leqq i \leqq N-1, \tag{8}$$

where

$$\mathbf{b}^{(i)} = \{b_1^{(i)}, b_2^{(i)}, \cdots, b_{n+1}^{(i)}\}^T, \tag{9}$$

$$\begin{aligned} A_{k,j} &= \varphi_{i+j}(t_i), \qquad k = 1, \\ &= (\varphi_{i+k}, \varphi'_{i+j})_i, \qquad 2 \leqq k \leqq n+1, 1 \leqq j \leqq n+1, \end{aligned} \tag{10}$$

$$\begin{aligned} c_k^{(i)}(\mathbf{b}^{(i)}) &= y_i, \qquad k = 1, \\ &= h \sum_{m=1}^{n} w_m f\left(\sigma_{i,m}, \sum_{j=1}^{n+1} b_j^{(i)} \varphi_{i+j}(\sigma_{i,m})\right) \varphi_{i+k}(\sigma_{i,m}), \qquad 2 \leqq k \leqq n+1. \end{aligned} \tag{11}$$

We consider only the cases where $\mathbf{A}$ is *nonsingular*. Certainly, $\mathbf{A}$ will be nonsingular when $\{\varphi_{i+k}\}_{k=2}^{n+1}$ span $\mathcal{P}_{n-1}$, the class of $(n - 1)$st degree polynomials. For then, $\mathbf{A}\mathbf{b}^{(i)} = \mathbf{0}$ implies $y(t_i) = 0$ and $(y', \varphi_{i+k})_i = 0$, $2 \leqq k \leqq n+1$, which, in turn, imply $y' \equiv 0$, $y \equiv 0$ and $\mathbf{b}^{(i)} = \mathbf{0}$. However, this condition is not necessary, since $\mathbf{A}$ is nonsingular in the case of the cubic $(n = 3)$ B-spline basis functions used for the computations given in Section 8, but $\{\varphi_{i+k}\}_{k=2}^{4}$ do not span $\mathcal{P}_2$. Since we may multiply (8) by $\mathbf{A}^{-1}$, our numerical method depends on the solution of

$$\mathbf{b}^{(i)} = \mathbf{A}^{-1}\mathbf{c}^{(i)}(\mathbf{b}^{(i)}), \qquad 0 \leqq i \leqq N-1. \tag{12}$$

3. Existence and Uniqueness of the Numerical Solution. Having let L denote the Lipschitz constant for f in $[t_0, t_N] \times R$, where $R \equiv (-\infty, \infty)$, we use the l_∞-norm to show that the right side of (12) is a contraction mapping on R^{n+1} when h is sufficiently small. Since

$$||\mathbf{A}^{-1}\mathbf{c}^{(i)}(\mathbf{b}) - \mathbf{A}^{-1}\mathbf{c}^{(i)}(\mathbf{b}^*)||_\infty \leqq ||\mathbf{A}^{-1}||_\infty \, ||\mathbf{c}^{(i)}(\mathbf{b}) - \mathbf{c}^{(i)}(\mathbf{b}^*)||_\infty$$

and

$$||\mathbf{c}^{(i)}(\mathbf{b}) - \mathbf{c}^{(i)}(\mathbf{b}^*)||_\infty \leqq hQ_1 L \, ||\mathbf{b} - \mathbf{b}^*||_\infty,$$

where

$$Q_1 = \max_{2 \leq k \leq n+1} \sum_{m=1}^{n} w_m \, |\varphi_{i+k}(\sigma_{i,m})| \sum_{j=1}^{n+1} |\varphi_{i+j}(\sigma_{i,m})|, \tag{13}$$

it is clear that

$$||\mathbf{A}^{-1}\mathbf{c}^{(i)}(\mathbf{b}) - \mathbf{A}^{-1}\mathbf{c}^{(i)}(\mathbf{b}^*)||_\infty \leqq hQ_2 \, ||\mathbf{b} - \mathbf{b}^*||_\infty,$$

where

$$Q_2 = Q_1 L \, ||\mathbf{A}^{-1}||_\infty. \tag{14}$$

Thus, we have a contraction mapping, and (12) has a unique solution which may be found by successive substitution when

$$h < Q_2^{-1}. \tag{15}$$

4. The Galerkin Method as a Collocation Method. We show here that the approximate solution $y(t)$ satisfies (1) at the quadrature points in each subinterval.

Using (11), we may write (12) as

$$(16) \qquad b_j^{(i)} = A_{j,1}^{-1} y_i + \sum_{m=1}^{n} \gamma_{j,m} f(\sigma_{i,m}, y(\sigma_{i,m})), \qquad 1 \leqq j \leqq n+1,$$

where

$$\gamma_{j,m} = h w_m \sum_{k=2}^{n+1} A_{j,k}^{-1} \varphi_{i+k}(\sigma_{i,m}).$$

Then, from substituting (16) into (3), we have at the quadrature points

$$(17) \qquad y'(\sigma_{i,k}) = \alpha_k y_i + \sum_{m=1}^{n} \beta_{m,k} f(\sigma_{i,m}, y(\sigma_{i,m})), \qquad 1 \leqq k \leqq n,$$

where

$$\alpha_k = \sum_{j=1}^{n+1} A_{j,1}^{-1} \varphi'_{i+j}(\sigma_{i,k})$$

and

$$\beta_{m,k} = \sum_{j=1}^{n+1} \gamma_{j,m} \varphi'_{i+j}(\sigma_{i,k}).$$

In the following, we make use of the fact that whenever f is independent of u and $f \in \mathcal{P}_{n-1}$, the exact solution $u \in \mathcal{P}_n$ and $y \equiv u$. This follows because the quadrature (6) is exact for $v \in \mathcal{P}_{2n-1}$, in this case $f\varphi_{i+k} \in \mathcal{P}_{2n-1}$, and the exact computation of $(f, \varphi_{i+k})_i$ means (8) is equivalent to (4)–(5). Since u satisfies (4)–(5) and y satisfies (8), they satisfy equivalent equations in this case and, by uniqueness, $u \equiv y$.

Let $q(t) \in \mathcal{P}_n$ be defined by $q(t_i) = 1$, $q'(\sigma_{i,k}) = 0$, $1 \leqq k \leqq n$, and let $f = q'$ so that $u' = f$, $u(t_i) = 1$ leads to $u = q = y$ on $[t_i, t_{i+1}]$. Substituting $y = q$ and $f = q'$ into (17) yields

$$(18) \qquad \alpha_k = 0, \qquad 1 \leqq k \leqq n.$$

Now for each r, $1 \leqq r \leqq n$, let $q_r(t) \in \mathcal{P}_n$ be defined by $q_r(t_i) = 0$, $q_r'(\sigma_{i,k}) = \delta_{r,k}$, $1 \leqq k \leqq n$, and let $f = q_r'$ and $u(t_i) = 0$ so that $u = q_r = y$. This time, substituting $y = q_r$ and $f = q_r'$ into (17) shows that

$$(19) \qquad \beta_{r,k} = \delta_{r,k}, \qquad 1 \leqq r, k \leqq n.$$

Consequently, (17) becomes the collocation equation

$$(20) \qquad y'(\sigma_{i,k}) = f(\sigma_{i,k}, y(\sigma_{i,k})), \qquad 1 \leqq k \leqq n,$$

showing that one-step collocation to (1) at the quadrature points by means of a continuous piecewise nth degree polynomial is equivalent to the Galerkin method.

Notice that the proof of this collocation property depends on the use of exactly n points in a quadrature formula (6) which is exact for $v \in \mathcal{P}_{2n-1}$. The proof would break down if (6) had more than n points or different weights and abscissae.

5. The Galerkin Method as an Implicit Runge-Kutta Method. Wright [26] has shown that any one-step collocation method is equivalent to some implicit Runge-Kutta method. Having already shown that the Galerkin method is equivalent to a certain one-step collocation method, we now derive the *particular* implicit Runge-Kutta method to which they are both equivalent. Of course, the Galerkin and collocation methods yield continuous approximations, so "equivalent" here means "matches the discrete values" of the Runge-Kutta approximation.

From (3) and (16), we have

$$y_{i+1} = \bar{\alpha} y_i + \sum_{m=1}^{n} \bar{\beta}_m f(\sigma_{i,m}, y(\sigma_{i,m})), \tag{21}$$

where

$$\bar{\alpha} = \sum_{j=1}^{n+1} A_{j,1}^{-1} \varphi_{i+j}(t_{i+1})$$

and

$$\bar{\beta}_m = \sum_{j=1}^{n+1} \gamma_{j,m} \varphi_{i+j}(t_{i+1}).$$

If we let $f = 0$, $u(t_i) = 1$ so that $u = 1 = y$, then substituting $y = 1$ and $f = 0$ into (21) produces

$$\bar{\alpha} = 1. \tag{22}$$

Next, for each r, $1 \leqq r \leqq n$, let $q_r(t) \in \mathcal{P}_n$ be defined as in Section 4. Now, substitution of $y = q_r$ and $f = q_r'$ into (21) leads to

$$q_r(t_{i+1}) = \bar{\beta}_r.$$

Since the n-point Gauss-Legendre formula (6) is exact for elements of $\mathcal{P}_{2n-1}$, we also have

$$q_r(t_{i+1}) = \int_{t_i}^{t_{i+1}} q_r'(t)\, dt = h \sum_{k=1}^{n} w_k q_r'(\sigma_{i,k}) = h w_r,$$

from which it follows that

$$\bar{\beta}_r = h w_r, \qquad 1 \leqq r \leqq n. \tag{23}$$

Together, (21)–(23) imply

$$y_{i+1} = y_i + h \sum_{m=1}^{n} w_m f(\sigma_{i,m}, y(\sigma_{i,m})), \tag{24}$$

and this is simply the implicit Runge-Kutta method based on the n-point Gauss-Legendre formula (6). Again, the proof of (24) depends on the fact that (6) is a Gauss-Legendre formula with exactly n points.

Thus, each of Butcher's implicit Runge-Kutta methods based on n-point Gauss-Legendre quadrature [2] has a corresponding "equivalent" Galerkin method using nth degree piecewise polynomials.

6. Error Bounds. In the following, a technique similar to that used by Shampine and Watts [21], [25] is employed to obtain asymptotic error bounds for the *discrete values* given by an implicit Runge-Kutta method. We view the Galerkin method as a discrete one-step method and use Henrici's theory [12, Chapter 2] of such methods. Continuous error bounds are then obtained from the discrete ones.

First, we need to define an increment function. Since, from (20), $y'(t)$ interpolates $f(t, y(t))$ at $\sigma_{i,k}$, $1 \leqq k \leqq n$, the Lagrangian representation for $y'(t)$ is

$$y'(t) = \sum_{k=1}^{n} l_k(t) f(\sigma_{i,k}, y(\sigma_{i,k})), \qquad t_i \leqq t \leqq t_{i+1}, \tag{25}$$

where

$$l_k(t) = \prod_{j=1; j \neq k}^{n} \frac{(t - \sigma_{i,j})}{(\sigma_{i,k} - \sigma_{i,j})}, \qquad 1 \leqq k \leqq n.$$

Integrating (25) leads to

$$y(t) = y_i + \sum_{k=1}^{n} f(\sigma_{i,k}, y(\sigma_{i,k})) \int_{t_i}^{t} l_k(s)\, ds, \qquad t_i \leqq t \leqq t_{i+1}. \tag{26}$$

Using (26), we now may write the Runge-Kutta form of the Galerkin method (24), in terms of an increment function Φ,

$$y_{i+1} = y_i + h\Phi(t_i, y_i; h), \qquad 0 \leqq i \leqq N - 1, \tag{27}$$

where Φ satisfies

$$\Phi(t_i, y_i; h) = \sum_{m=1}^{n} w_m g_m(t_i, y_i; h) \tag{28}$$

and

$$\begin{aligned} g_m(t_i, y_i; h) &= f(\sigma_{i,m}, y(\sigma_{i,m})) \\ &= f\left(t_i + \theta_m h, y_i + \sum_{k=1}^{n} g_k(t_i, y_i; h) \int_{t_i}^{t_i + \theta_m h} l_k(s)\, ds\right), \qquad 1 \leqq m \leqq n. \end{aligned} \tag{29}$$

In order for Henrici's theory to apply, we must show that Φ is Lipschitz continuous with respect to y in $\Omega \equiv [t_0, t_N] \times R \times [0, h_0]$. If, for any i, $0 \leqq i \leqq N - 1$, and any $y_i^* \in R$, $y^*(t)$ is the Galerkin approximate solution to $u' = f(t, u)$, $u(t_i) = y_i^*$, $t_i \leqq t \leqq t_{i+1}$, then (26) holds for y^*

$$y^*(t) = y_i^* + \sum_{k=1}^{n} f(\sigma_{i,k}, y^*(\sigma_{i,k})) \int_{t_i}^{t} l_k(s)\, ds, \qquad t_i \leqq t \leqq t_{i+1}. \tag{26'}$$

Letting B_0 be a constant such that

$$\sum_{k=1}^{n} \max_{t_i \leqq t \leqq t_{i+1}} \left| \int_{t_i}^{t} l_k(s)\, ds \right| \leqq hB_0, \qquad 0 \leqq i \leqq N - 1, \tag{30}$$

and subtracting (26) and (26′) leads to

$$\max_{t_i \leqq t \leqq t_{i+1}} |y(t) - y^*(t)| \leqq \frac{1}{1 - h_0 B_0 L} |y_i - y_i^*|, \qquad 0 \leqq i \leqq N - 1, \tag{31}$$

where $0 \leqq h \leqq h_0 < (B_0 L)^{-1}$. The Lipschitz condition then follows from (28), (29) and (31) since, for $0 \leqq h \leqq h_0$ and $0 \leqq i \leqq N - 1$,

$$
\begin{aligned}
|\Phi(t_i, y_i; h) - \Phi(t_i, y_i^*; h)| &\leq \sum_{m=1}^{n} w_m \, |g_m(t_i, y_i; h) - g_m(t_i, y_i^*; h)| \\
&\leq L \sum_{m=1}^{n} w_m \, |y(\sigma_{i,m}) - y^*(\sigma_{i,m})| \qquad (32) \\
&\leq \frac{L}{1 - h_0 B_0 L} |y_i - y_i^*|,
\end{aligned}
$$

where $\sum_{m=1}^{n} w_m = 1$.

Now, we may prove

THEOREM 1. *Assume that $f(t, x) \in C^{2n}$ in $[t_0, t_N] \times R$ so that $u(t) \in C^{2n+1}[t_0, t_N]$, and denote by L the Lipschitz constant for f in this region. Let the Galerkin method be defined as in Section* 2 *for some piecewise polynomial basis functions of degree $n \geq 1$ and the n-point Gauss-Legendre quadrature formula* (6). *If Q_2 and B_0 are defined by* (14) *and* (30), *respectively, and $0 < h \leq h_0$ where $0 < h_0 < \min(Q_2^{-1}, (B_0L)^{-1})$, then there exists a constant M such that*

$$
|u_i - y_i| \leq M h^{2n}, \qquad 0 \leq i \leq N. \tag{33}
$$

Proof. The local truncation error τ_i is defined from (24) by

$$
u_{i+1} = u_i + h \sum_{m=1}^{n} w_m f(\sigma_{i,m}, u(\sigma_{i,m})) + \tau_i .
$$

Thus,

$$
\tau_i = \int_{t_i}^{t_{i+1}} f(t, u(t)) \, dt - h \sum_{m=1}^{n} w_m f(\sigma_{i,m}, u(\sigma_{i,m}))
$$

and, from (6), $|\tau_i| \leq K h^{2n+1}$, where K is a constant depending on the maximum of $u^{(2n+1)}(t)$ on $[t_0, t_N]$. The bound (33) follows immediately from Henrici's Theorem 2.2 [12]. Q.E.D.

The discrete error bounds (33) agree with those for Butcher's methods [2].

We obtain continuous error bounds in

THEOREM 2. *Let the hypotheses of Theorem* 1 *hold. Then there exist constants E_j, $0 \leq j \leq n$, such that*

$$
\max_{t_0 \leq t \leq t_N} |u(t) - y(t)| \leq E_0 h^{n+1}, \tag{34}
$$

and

$$
\max_{t_i \leq t \leq t_{i+1}} |u^{(j)}(t) - y^{(j)}(t)| \leq E_j h^{n-j+1}, \qquad 1 \leq j \leq n, \; 0 \leq i \leq N-1. \tag{35}
$$

Proof. We write $u(t)$ in the same form as $y(t)$ in (26) by using the n-point Lagrangian quadrature found there

$$
\begin{aligned}
u(t) &= u_i + \int_{t_i}^{t} f(s, u(s)) \, ds \\
&= u_i + \sum_{k=1}^{n} f(\sigma_{i,k}, u(\sigma_{i,k})) \int_{t_i}^{t} l_k(s) \, ds + R_n(t), \qquad t_i \leq t \leq t_{i+1},
\end{aligned} \tag{36}
$$

where $R_n(t) = O(h^{n+1})$. Subtracting from (26), we find that

$$\max_{t_i \leq t \leq t_{i+1}} |u(t) - y(t)| \leqq \frac{1}{1 - h_0 B_0 L} |u_i - y_i| + O(h^{n+1}), \qquad 0 \leqq i \leqq N - 1;$$

and (34) follows from (33). If we differentiate (26) and (36) j times using $R_n^{(j)}(t) = O(h^{n-j+1})$ and subtract, we can show that

$$\max_{t_i \leq t \leq t_{i+1}} |u^{(j)}(t) - y^{(j)}(t)| \leqq LB_j \max_{1 \leq k \leq n} |u(\sigma_{i,k}) - y(\sigma_{i,k})| + O(h^{n-j+1}),$$

for $1 \leqq j \leqq n$, $0 \leqq i \leqq N - 1$, where

$$\sum_{k=1}^{n} \max_{t_i \leq t \leq t_{i+1}} |l_k^{(j-1)}(t)| \leqq B_j.$$

Then (35) follows from (34). Q.E.D.

7. A-Stability of the Galerkin Methods. Dahlquist [8] defines A-stability as follows.

Definition. A k-step method is called *A-stable*, if all its solutions tend to zero, as $i \to \infty$, when the method is applied with fixed positive h to any differential equation of the form $u' = \lambda u$, where λ is a complex constant with negative real part.

Butcher's implicit Runge-Kutta methods based on Gauss-Legendre quadrature [2] have been shown by Ehle [10] to be A-stable. Ehle observed that the n-stage method, applied to $u' = \lambda u$, yields $y_{i+1} = P_{nn}(\lambda h) y_i$, where $P_{nn}(\lambda h)$ is the nth diagonal Padé rational approximation to $\exp(\lambda h)$. A-stability follows from the fact that $|P_{nn}(\lambda h)| < 1$ for Re $(\lambda h) < 0$. Our Galerkin methods, which from (24) give discrete values y_i identical to those of Butcher's methods [2], are therefore A-stable.

We should remark that Axelsson [1] has used similar properties of subdiagonal and diagonal Padé rational approximations to prove A-stability for implicit Runge-Kutta methods based on Radau and Lobatto quadratures. It is natural then to ask whether a Galerkin method which uses these quadratures rather than Gauss-Legendre would yield corresponding "equivalent" methods. The answer is no. If (6) were an n-point Radau formula with $\sigma_{i,n} = t_{i+1}$, it would be exact only for $v \in \mathcal{P}_{2n-2}$. The quadratures for $(f, \varphi_{i+k})_i$ would not be exact for $f \in \mathcal{P}_{n-1}$, y would not be exact for $u \in \mathcal{P}_n$, (24) would not hold, and the order of the Galerkin method would be $O(h^{n-1})$, whereas Axelsson [1] and Butcher [3] have shown that an n-stage implicit Runge-Kutta method based on Radau quadrature has the order $O(h^{2n-1})$. Similar results are true of Lobatto quadrature.

8. Numerical Examples. In this section, we give numerical results of an A-stable piecewise cubic ($n = 3$) Galerkin scheme of order 6. We have employed Schoenberg's [20] cubic B-spline basis functions where φ_{i+j} has its support on $[t_{i+j-4}, t_{i+j}]$. The calculations were performed on a CDC 6600, which has about 14 decimal digits, using a successive substitution iteration at each step to solve (12).

First, we consider problems for single equations.

Problem 1. $u' = -2tu^2$, $u(0) = 1$, $u(t) = 1/(1 + t^2)$, $0 \leqq t \leqq 1$.

Problem 2. $u' = 1/(1 + \tan^2 u)$, $u(0) = 0$, $u(t) = \arctan t$, $0 \leqq t \leqq 1$.

Problem 3. $u' = u - (2t/u)$, $u(0) = 1$, $u(t) = (2t + 1)^{1/2}$, $0 \leqq t \leqq 1$.
Problem 4. $u' = u$, $u(0) = 1$, $u(t) = e^t$, $0 \leqq t \leqq 10$.
Several uniform meshes are used for each problem. Tables 1–4 are designed to

TABLE 1

Error Norms for Problem 1

h	$\|\|e(t; h)\|\|'$	$\|\|e'(t; h)\|\|'$	$\|\|e''(t; h)\|\|'$	$\|\|e'''(t; h)\|\|'$
1	$3.55(10)^{-4}$	$7.94(10)^{-2}$	$8.20(10)^{-1}$	$3.49(10)^{0}$
2^{-1}	$8.54(10)^{-6}$ (5.38)	$1.30(10)^{-2}$ (2.61)	$3.36(10)^{-1}$ (1.29)	$4.10(10)^{0}$ (−0.23)
2^{-2}	$1.18(10)^{-7}$ (6.18)	$2.65(10)^{-3}$ (2.30)	$1.29(10)^{-1}$ (1.38)	$2.71(10)^{0}$ (0.60)
2^{-3}	$1.79(10)^{-9}$ (6.04)	$3.75(10)^{-4}$ (2.82)	$3.61(10)^{-2}$ (1.84)	$1.46(10)^{0}$ (0.89)
2^{-4}	$2.81(10)^{-11}$ (6.00)	$4.83(10)^{-5}$ (2.96)	$9.29(10)^{-3}$ (1.96)	$7.45(10)^{-1}$ (0.97)
2^{-5}	$3.45(10)^{-13}$ (6.35)	$6.09(10)^{-6}$ (2.99)	$2.34(10)^{-3}$ (1.99)	$3.74(10)^{-1}$ (0.99)
2^{-6}	$1.85(10)^{-13}$ (0.90)	$7.62(10)^{-7}$ (3.00)	$5.86(10)^{-4}$ (2.00)	$1.87(10)^{-1}$ (1.00)

TABLE 2

Error Norms for Problem 2

h	$\|\|e(t; h)\|\|'$	$\|\|e'(t; h)\|\|'$	$\|\|e''(t; h)\|\|'$	$\|\|e'''(t; h)\|\|'$
1	$2.48(10)^{-5}$	$3.04(10)^{-2}$	$3.62(10)^{-1}$	$1.62(10)^{0}$
2^{-1}	$1.28(10)^{-7}$ (7.60)	$4.76(10)^{-3}$ (2.67)	$1.11(10)^{-1}$ (1.70)	$6.61(10)^{-1}$ (1.29)
2^{-2}	$5.79(10)^{-9}$ (4.46)	$5.88(10)^{-4}$ (3.02)	$2.84(10)^{-2}$ (1.97)	$5.70(10)^{-1}$ (0.21)
2^{-3}	$8.62(10)^{-11}$ (6.07)	$7.64(10)^{-5}$ (2.94)	$7.30(10)^{-3}$ (1.96)	$2.85(10)^{-1}$ (1.00)
2^{-4}	$1.34(10)^{-12}$ (6.01)	$9.48(10)^{-6}$ (3.01)	$1.82(10)^{-3}$ (2.01)	$1.46(10)^{-1}$ (0.97)
2^{-5}	$9.24(10)^{-14}$ (3.86)	$1.19(10)^{-6}$ (2.99)	$4.56(10)^{-4}$ (1.99)	$7.29(10)^{-2}$ (1.00)
2^{-6}	$1.49(10)^{-13}$ (−0.69)	$1.49(10)^{-7}$ (3.00)	$1.14(10)^{-4}$ (2.00)	$3.64(10)^{-2}$ (1.00)

TABLE 3

Error Norms for Problem 3

h	$\|\|e(t; h)\|\|'$	$\|\|e'(t; h)\|\|'$	$\|\|e''(t; h)\|\|'$	$\|\|e'''(t; h)\|\|'$
1	$7.08(10)^{-4}$	$2.57(10)^{-2}$	$3.24(10)^{-1}$	$2.43(10)^{0}$
2^{-1}	$2.22(10)^{-5}$ (4.99)	$6.03(10)^{-3}$ (2.09)	$1.50(10)^{-1}$ (1.11)	$1.87(10)^{0}$ (0.37)
2^{-2}	$4.67(10)^{-7}$ (5.57)	$1.14(10)^{-3}$ (2.40)	$5.58(10)^{-2}$ (1.42)	$1.26(10)^{0}$ (0.57)
2^{-3}	$8.05(10)^{-9}$ (5.86)	$1.83(10)^{-4}$ (2.64)	$1.77(10)^{-2}$ (1.65)	$7.55(10)^{-1}$ (0.74)
2^{-4}	$1.30(10)^{-10}$ (5.96)	$2.63(10)^{-5}$ (2.80)	$5.07(10)^{-3}$ (1.81)	$4.19(10)^{-1}$ (0.85)
2^{-5}	$2.73(10)^{-12}$ (5.57)	$3.53(10)^{-6}$ (2.89)	$1.36(10)^{-3}$ (1.90)	$2.21(10)^{-1}$ (0.92)
2^{-6}	$1.48(10)^{-12}$ (0.88)	$4.59(10)^{-7}$ (2.95)	$3.53(10)^{-4}$ (1.95)	$1.14(10)^{-1}$ (0.96)

TABLE 4

Error Norms for Problem 4

h	$\|\|e(t;h)\|\|'$		$\|\|e'(t;h)\|\|'$		$\|\|e''(t;h)\|\|'$		$\|\|e'''(t;h)\|\|'$	
1	$2.27(10)^{0}$		$1.15(10)^{2}$		$1.48(10)^{3}$		$5.59(10)^{3}$	
2^{-1}	$3.45(10)^{-2}$	(6.04)	$1.80(10)^{1}$	(2.67)	$4.49(10)^{2}$	(1.72)	$3.90(10)^{3}$	(0.52)
2^{-2}	$5.35(10)^{-4}$	(6.01)	$2.54(10)^{0}$	(2.83)	$1.24(10)^{2}$	(1.86)	$2.31(10)^{3}$	(0.75)
2^{-3}	$8.31(10)^{-6}$	(6.01)	$3.37(10)^{-1}$	(2.92)	$3.27(10)^{1}$	(1.93)	$1.26(10)^{3}$	(0.88)
2^{-4}	$9.32(10)^{-8}$	(6.48)	$4.34(10)^{-2}$	(2.96)	$8.38(10)^{0}$	(1.96)	$6.59(10)^{2}$	(0.94)
2^{-5}	$6.23(10)^{-8}$	(0.58)	$5.52(10)^{-3}$	(2.98)	$2.12(10)^{0}$	(1.98)	$3.37(10)^{2}$	(0.97)
2^{-6}	$1.27(10)^{-7}$	(−1.03)	$6.95(10)^{-4}$	(2.99)	$5.34(10)^{-1}$	(1.99)	$1.70(10)^{2}$	(0.98)

illustrate the $O(h^6)$ mesh point accuracy of Theorem 1 as well as the $O(h^3)$, $O(h^2)$ and $O(h)$ accuracies of the first three derivatives predicted by Theorem 2. The tables give the *discrete error norms* for $y(t; h)$ and its first three derivatives

$$||e^{(j)}(t;h)||' = \max_{0\leq i\leq N} |e^{(j)}(t_{i\pm};h)|, \qquad 0 \leqq j \leqq 3, \tag{37}$$

where $e = u - y$ and also in parentheses the *computed orders of accuracy*, based on successive mesh sizes h_1, h_2,

$$\omega_j = \frac{\log[||e^{(j)}(t;h_1)||'/||e^{(j)}(t;h_2)||']}{\log(h_1/h_2)}, \tag{38}$$

i.e., $||e^{(j)}(t;h)||' \approx O(h^{\omega_j})$, $0 \leqq j \leqq 3$.

Next, we present in Table 5 absolute errors $e(t; h)$ and relative errors $e(t; h)/u(t)$

TABLE 5

Absolute and Relative Errors for Problem 5

t	$e(t;1)$	$e(t;1)/u(t)$	$e(t;0.5)$	$e(t;0.5)/u(t)$
1	$3.79(10)^{-6}$	$1.03(10)^{-5}$	$5.76(10)^{-8}$	$1.57(10)^{-7}$
10	$4.68(10)^{-9}$	$1.03(10)^{-4}$	$7.11(10)^{-11}$	$1.57(10)^{-6}$
20	$4.25(10)^{-13}$	$2.06(10)^{-4}$	$6.45(10)^{-15}$	$3.13(10)^{-6}$
40	$1.75(10)^{-21}$	$4.12(10)^{-4}$	$2.67(10)^{-23}$	$6.26(10)^{-6}$
60	$5.42(10)^{-30}$	$6.19(10)^{-4}$	$8.22(10)^{-32}$	$9.39(10)^{-6}$
80	$1.49(10)^{-38}$	$8.25(10)^{-4}$	$2.26(10)^{-40}$	$1.25(10)^{-5}$
100	$3.83(10)^{-47}$	$1.03(10)^{-3}$	$5.83(10)^{-49}$	$1.57(10)^{-5}$

at selected points t_i for $h = 1$ and 0.5 in

Problem 5. $u' = -u$, $u(0) = 1$, $u(t) = e^{-t}$, $0 \leqq t \leqq 100$,

in order to illustrate the stability of the method.

Finally, we give in Tables 6 and 7 the results of the application of our method to

TABLE 6

Error Norms for $e_1(t; h)$ of Problem 6

h	$\|\|e_1(t; h)\|\|'$	$\|\|e_1'(t; h)\|\|'$	$\|\|e_1''(t; h)\|\|'$	$\|\|e_1'''(t; h)\|\|'$
1	$1.97(10)^{-3}$	$1.82(10)^{-2}$	$1.77(10)^{-1}$	$7.44(10)^{-1}$
2^{-1}	$2.91(10)^{-5}$ (6.08)	$2.53(10)^{-3}$ (2.85)	$5.43(10)^{-2}$ (1.71)	$4.95(10)^{-1}$ (0.59)
2^{-2}	$4.50(10)^{-7}$ (6.02)	$3.33(10)^{-4}$ (2.92)	$1.52(10)^{-2}$ (1.84)	$2.89(10)^{-1}$ (0.77)
2^{-3}	$7.01(10)^{-9}$ (6.00)	$4.29(10)^{-5}$ (2.96)	$4.01(10)^{-3}$ (1.92)	$1.57(10)^{-1}$ (0.88)
2^{-4}	$1.08(10)^{-10}$ (6.02)	$5.45(10)^{-6}$ (2.98)	$1.03(10)^{-3}$ (1.96)	$8.16(10)^{-2}$ (0.94)
2^{-5}	$2.19(10)^{-12}$ (5.62)	$6.86(10)^{-7}$ (2.99)	$2.62(10)^{-4}$ (1.98)	$4.16(10)^{-2}$ (0.97)
2^{-6}	$2.69(10)^{-12}$ (−1.61)	$8.61(10)^{-8}$ (2.99)	$6.59(10)^{-5}$ (1.99)	$2.10(10)^{-2}$ (0.99)

TABLE 7

Error Norms for $e_2(t; h)$ of Problem 6

h	$\|\|e_2(t; h)\|\|'$	$\|\|e_2'(t; h)\|\|'$	$\|\|e_2''(t; h)\|\|'$	$\|\|e_2'''(t; h)\|\|'$
1	$2.58(10)^{-4}$	$7.26(10)^{-3}$	$7.69(10)^{-2}$	$3.96(10)^{-1}$
2^{-1}	$3.82(10)^{-6}$ (6.08)	$9.40(10)^{-4}$ (2.95)	$2.17(10)^{-2}$ (1.82)	$2.21(10)^{-1}$ (0.84)
2^{-2}	$5.91(10)^{-8}$ (6.02)	$1.23(10)^{-4}$ (2.93)	$5.82(10)^{-3}$ (1.90)	$1.18(10)^{-1}$ (0.91)
2^{-3}	$9.20(10)^{-10}$ (6.00)	$1.58(10)^{-5}$ (2.96)	$1.51(10)^{-3}$ (1.95)	$6.06(10)^{-2}$ (0.96)
2^{-4}	$1.42(10)^{-11}$ (6.02)	$2.00(10)^{-6}$ (2.98)	$3.84(10)^{-4}$ (1.97)	$3.08(10)^{-2}$ (0.98)
2^{-5}	$2.13(10)^{-13}$ (6.06)	$2.52(10)^{-7}$ (2.99)	$9.68(10)^{-5}$ (1.99)	$1.55(10)^{-2}$ (0.99)
2^{-6}	$7.27(10)^{-13}$ (−1.77)	$3.17(10)^{-8}$ (2.99)	$2.43(10)^{-5}$ (1.99)	$7.78(10)^{-3}$ (0.99)

the system of equations in

Problem 6. $u_1' = u_1^2 u_2$, $u_2' = -1/u_1$, $u_1(0) = 1$, $u_2(0) = 1$, $u_1 = e^t$ $u_2 = e^{-t}$, $0 \leqq t \leqq 1$.

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